Cox rings of surface quotient singularities

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Let X be a normal (pre)variety with free finitely generated class group. The Cox ring of X is

$$\mathrm{Cox}(X) = \bigoplus_{[D]\in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_D(X)).$$

Our task is to describe the structure of $\mathrm{Cox}(\widetilde{X})$ where $\widetilde{X}$ is a (minimal) resolution of a surface quotient singularity $X \simeq \mathbb{C}/G$ for a finite linear group G.

Geometric point of view: assume that $\mathrm{Cox}(X)$ is finitely generated; then on $\mathrm{Spec}(\mathrm{Cox}(\widetilde{X}))$ there acts the torus $T_X = \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{C}^*)$ and X, $\widetilde{X}$ and other resolutions are GIT quotients of $\mathrm{Spec}(\mathrm{Cox}(\widetilde{X}))$ under this action. Thus the Cox ring contains a lot of information about the singularity and its resolutions.

Cox rings of Du Val singularities

Du Val singularities are the quotients of $\mathbb{C}^2$ by finite subgroups of $SL(2,\mathbb{C})$: A_n for the cyclic group $\mathbb{Z}_n$, D_n for binary dihedral group BD_n and E_6, E_7, E_8 for binary tetrahedral BT, octahedral BO and icosahedral BI group respectively.

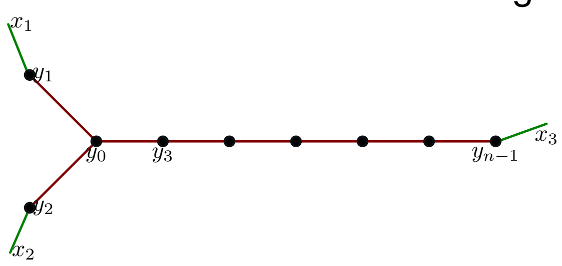
Classical fact: $\mathrm{Cox}(\widetilde{A_n})$ is a polynomial ring.

Theorem (from [3]). The spectrum of the Cox ring for D_n and E_6, E_7, E_8 is a hypersurface defined by a trinomial equation in $\mathbb{C}^{n+3}$.

For example, for D_n it is

$$x_1^2 y_1 + x_2^2 y_2 + y_3 y_4^2 \cdots y_{n-1}^{n-3} x_3^{n-2} = 0.$$

The relation can be read out from the diagram:



the exponent of y_i is the distance from the central point; the x_i's are added 'tails' of the diagram.

This is the result of [3]. Their method uses the equation defining the singularity as a surface in $\mathbb{C}^3$.

Problem: very hard to generalize!

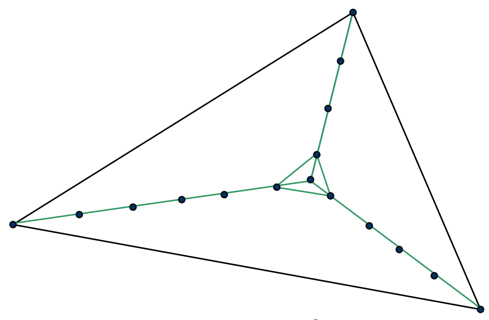
Our approach (for $G < GL(2,\mathbb{C})$)

The starting point is understanding the structure of the minimal resolution of the singularity. This is the result of Brieskorn, [2]: the curves in the special fibre form a tree similar to the Dynkin diagram, but not all the curves are (-2)-curves. We use it to define the action of T_X on $\mathbb{C}^{n+3}$ where $n = rk(\mathrm{Cl}(X))$.

Next, we describe a big open set in $\mathrm{Spec}(\mathrm{Cox}(\widetilde{X}))$, a generalization of the universal torsor (the relative spectrum of the Cox sheaf). It is expected to be some quotient of a torus bundle. It turns out to be a quotient of a trivial T_X bundle over $\mathbb{C}/[G,G]$ by an (induced) action of $Ab(G) = G/[G,G]$.

Then we look for a compactification of this set. For $G < GL(2,\mathbb{C})$ it is a hypersurface S in $\mathbb{C}^{n+3}$, its equation can be read out of the the diagram of the resolution, but not as simply as in the case of Du Val singularities – Hirzebruch-Jung continued fractions are involved.

Using the matrix of intersection numbers of the curves in the exceptional fibre we define an action of T_X on $\mathbb{C}^{n+3}$ and S. We need to understand the quotient S/T_X and prove that it is a resolution of X. The main point of the proof is the smoothness of the quotient. We investigate S/T_X by looking at it as a hypersurface in the toric variety $\mathbb{C}^{n+3}/T_X$. We can describe $\mathbb{C}^{n+3}/T_X$ up to small modifications – the set of rays of its fan can be well understood. Its section looks as in the picture below (only the faces which have to be in the fan are drawn):



We choose some model of $\mathbb{C}^{n+3}/T_X$, analyze the equation of S/T_X on each affine piece and prove the smoothness locally.

Finally, we use theorems [1, 6.4.3, 6.4.4]– a characterization of the $\mathrm{Spec}(\mathrm{Cox}(\widetilde{X}))$ in terms of the quotient by a torus action.

The result: Cox rings for $\mathbb{C}/G$ for all finite $G < GL(2,\mathbb{C})$ computed; they are finitely generated.

Currently we work on generalizing these results so that they can be applied to to the 4-dimensional singularities. We would like to use the Cox ring to look for new examples of symplectic resolutions of 4-dimensional quotient singularities. This is joint work with J. Wiśniewski.

Our setting is in fact a special case of varieties with torus action of complexity one, so the problem can be also considered in connection with [4].

References

[1] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface, *Cox Rings*, arxiv (2011).

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[3] Laura Facchini, Víctor González-Alonso, and Michał Lasoń, *Cox rings of Du Val singularities*, Le Matematiche **66** (2011), 115–136.

[4] Jürgen Hausen and Hendrik Süß, *The Cox ring of an algebraic variety with torus action*, Advances in Mathematics **225** (2010), no. 2, 977–1012.