Explicit period maps and the absence of stable Schottky relations

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Explicit period matrices

Let us denote, as usual, with M_g the moduli space of smooth genus g curves, and with A_g the moduli space of principal polarised abelian g-fold. The Siegel upper space is the space of g-by-g complex symmetric matrices whose imaginary part is positive definite. A_g is the quotient of the Siegel upper space by a discrete group, so the entries of the matrices are local coordinates for A_g.

We are interested in the image of M_g in A_g via the Jacobian morphism. Locally, the Jacobian morphism maps a curve to its period matrix. A one-parameter family of curves is equivalent to an arc on M_g, therefore, a formula for the period matrix as function of the parameter is an **explicit parametrisation of an arc on** M_g.

The literature about period matrices is huge, but it is quite rare to find a formula as the one described here. A good source about this subject is the third chapter of [Fay70]. The author constructs some one parameter families of curves and he computes the first order term of the period matrix. The key ingredients of this computation are the Riemann bilinear relations for differentials of the second kind.

The Satake compactification

The Satake compactification A_g^S is the normal compactification of A_g constructed using modular forms. It comes with a stratification $A_g^S = A_g \sqcup A_{g-1}^S$ and an ample line bundle $\mathscr{L}_g$. The sections of $\mathscr{L}_g^{\otimes k}$ are all modular forms of weight k, and the restriction of this line bundle from A_g^S to A_{g-1}^S is exactly $\mathscr{L}_{g-1}^{\otimes k}$.

Stable modular forms

A stable modular form is the datum of a section of $\mathscr{L}_g^{\otimes k}$ for every g compatible with the restriction. Quite surprisingly, given an even unimodular lattice Γ, one can define a stable modular form θ_Γ, called the Theta series associated to Γ; moreover, the ring of stable modular forms is the polynomial ring in the θ_Γ's, with Γ irreducible.

An example: the Schottky form

There are exactly two even unimodular lattices of rank 16, and the difference of the associated Theta series is the so-called Schottky form. It is well known that it is zero for g smaller than or equal to 3, it is the equation for M_4 in A_4 and it vanishes on the hyperelliptic locus for every g. More recently, it has been proved that it is not zero on M_5, but it cuts out the trigonal locus (i.e. 3-sheeted covers of $\mathbb{P}^1$), see [GM11].

The absence of stable Schottky relations

Natural objects to look for are the stable Schottky relations, these are stable modular forms vanishing on M_g for every g. In other words, they are **stable equations for the Jacobian locus inside** A_g. They are related to the singularities of the Satake compactification M_g^S of M_g. This compactification is just the closure of M_g inside A_g^S and is covered with strata: each stratum isomorphic, via the Jacobian morphism, to a product $M_{g_1} \times \cdots \times M_{g_s}$ with $\sum g_i \leq g$. Inside A_{g+m}^S, as sets, $A_g \cap M_{g+m}^S = M_g$. The main result we can prove using explicit period matrices is the following:

Theorem ([CSB11])**.** *The intersection of A_g and M_{g+m}^S contains the m-th infinitesimal neighbourhood of M_g inside A_g, so* ***stable Schottky relations do not exist****.*

One of the goals of our research is to describe the singularities of M_g^S at $M_{g-1} \times M_1$.

Stable equations for the hyperelliptic locus

We are trying to work out the same analysis for the hyperelliptic locus. So far, we can prove the following result:

Theorem (C. 2012)**.** *Let Γ and Λ be two even unimodular lattices of rank 32 without vectors of norm 2, then $\theta_\Gamma - \theta_\Lambda$* ***is a stable equation for the hyperelliptic locus****.*

This result follows from an explicit analysis of the tangent space of the Satake compactification of the hyperelliptic locus, and a better understanding of the singularities could lead to significant refinements. It is known that there exist at least 10 millions of lattices satisfying the hypothesis of the theorem.

One could also try to find stable equations for the n-gonal locus (i.e. n-sheeted covers of $\mathbb{P}^1$), but it is not clear how to compute explicitly period matrices for n-gonal families: the main complication with respect to the hyperelliptic case is the absence of automorphisms of the projection to $\mathbb{P}^1$.

Stable divisors

We can give another interpretation of the absence of stable Schottky relations: any linear combination of Theta series defines a stable divisor on M_g, for all sufficiently large g. The point is that, so far, we do not have a geometric interpretation of these divisors.

Higher dimensional analogues

So far we have been dealing with curves, but period matrices are the one-dimensional version of period maps. As for period matrices, there is lot of literature about period maps, but it is quite hard to find explicit computations. We are interested in finding some formulae for the period maps of degenerating families; in order to do this one should, probably, figure out a higher dimensional analogue of the Riemann bilinear relations.

References

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