Categorical geometry : Hodge structures and K-theory

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The 'classical' picture

LET X be a smooth projective variety over $\mathbb{C}$. Then we can consider X as a complex variety in order to use the standard tools from Hodge theory to study the analytic topology of X. The usual package includes for all $k \geq 0$ a Hodge structure of weight k on the De Rham cohomology group $\mathrm{H}^k_{DR}(X, \mathbb{C})$. This Hodge structure is given by

- The Betti cohomology $\mathrm{H}^k_B(X, \mathbb{Z})$, $k \geq 0$, with integer coefficients, via the map
$$\mathrm{H}^k_B(X, \mathbb{Z}) \longrightarrow \mathrm{H}^k_B(X, \mathbb{C}) \simeq \mathrm{H}^k_{DR}(X, \mathbb{C})$$
whose image is a lattice for all $k \geq 0$,
- The Hodge filtration on $\mathrm{H}^k_{DR}(X, \mathbb{C})$ for all $k \geq 0$, given by the natural filtration (by truncation) on the De Rham complex.
- The symmetry $\overline{\mathrm{H}^q(X, \Omega^p_X)} = \mathrm{H}^p(X, \Omega^q_X)$.

The algebraic part of the structure, ie the Hodge filtration, is equivalent to the degeneration at the first stage of a certain Hodge-to-De Rham spectral sequence

$$E_1^{pq} = \mathrm{H}^q(X, \Omega^p_X) \Longrightarrow \mathrm{H}^{p+q}_{DR}(X, \mathbb{C}).$$

These Hodge structures on the cohomology of smooth complex projective varieties revealed themselves to be a powerful tool in the study of the geometry of such varieties. Among other things we can quote their relation to Kähler geometry, intersection theory and the study of algebraic cycles. Following the *non-commutative* or *categorical philosophy* developped in the work of Kontsevich (see Marco Robalo's poster), one can study an algebraic variety from the derived dg-category of sheaves of some sort on it. And more generally, one can think of any dg-category as the dg-category of sheaves on an hypothetical space. Hence it is a natural and important question to ask for similar structures on any nice dg-category coming or not from an algebraic variety.

The categorical picture

THE categorical counterpart of De Rham cohomology is given by the periodic cyclic homology of dg-categories and the Hodge cohomology of a variety corresponds to the Hochschild homology of dg-categories. Analog to the Hodge-to-De Rham spectral sequence, there is for any dg-category T a spectral sequence

$$\mathrm{HH}_\bullet(T)[u^{-1}] \Longrightarrow \mathrm{HP}_\bullet(T).$$

(multiplication by u is the circle action). A conjecture by Kontsevich-Soibelman (see [4]) says that this spectral sequence degenerates at the first stage for any smooth and proper dg-algebra over a field of characteristic zero. Kaledin proved this degeneration conjecture for the bounded below dg-algebras using Deligne-Illusie method of reduction to positive characteristic (see [2]). The degeneration conjecture is a central statement in the growing subject of categorical geometry, which is related to the hypothetic existence of a non-commutative Hodge structure in the sense of [3] on $\mathrm{HP}(T)$. This leads to the following famous problem :

Question : What is the categorical counterpart of the Betti cohomology ?

My work is to promulgate a candidate known as **topological K-theory** of dg-categories. Looking in topology, one can find a first evidence for this choice in the rational isomorphism given by the Chern character map

$$K^0(X) \otimes \mathbb{Q} \simeq \mathrm{H}^{ev}_B(X, \mathbb{Q}),$$

for X a nice enough space. We emphasize that the Betti cohomology has a transcendental nature. Therefore, the topological K-theory functor K^{top} is defined (here) only over complex numbers and using a topological realization functor of stacks over $\mathbb{C}$ extending the usual analytic space functor $Aff/\mathbb{C} \longrightarrow Top$. The construction of the topological K-theory is inspired and parallel to the construction of Friedlander-Walker's semi-topological K-theory of complex quasi-projective varieties ([1]) which interpolates between the algebraic and the topological K-theory. If T is a $\mathbb{C}$-dg-category, one possible definition is

$$K^{top}(T) := |\mathcal{M}_T|[\beta^{-1}],$$

where $|\mathcal{M}_T|$ is the spectrum associated to the realization of the moduli stack $\mathcal{M}_T$ which classifies objects in T (see [5]), and β stands for the Bott element. I have made explicit the calculation $K^{top}(\mathbb{C}) \simeq \mathbf{bu}$. An on-going result is the following

Theorem. *For any smooth and proper $\mathbb{C}$-dg-category T there exists a topological Chern character*

$$Ch^{top} : K^{top}(T) \longrightarrow \mathrm{HP}(T)$$

which is a factorisation of the already known algebraic Chern character, obtaining a commutative square

$$\begin{array}{ccc} K^{alg}(T) & \xrightarrow{Ch} & \mathrm{HC}^-(T) \\ \downarrow & & \downarrow \\ K^{top}(T) & \xrightarrow{Ch^{top}} & \mathrm{HP}(T) \end{array}$$

References

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