The Tautological Ring of the Universal Curve

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Background

- The *universal curve*, $\mathcal{C}_g$ or $\mathcal{M}_{g,1}$, (over the moduli space of smooth curves of genus $g \geq 2$ over the algebraically closed field k, $\mathcal{M}_g$) is the moduli space of smooth curves of genus g with one marked point.
- One can define the Chow ring of $\mathcal{C}_g$, $A^*(\mathcal{C}_g)$. However, one has to work with rational coefficients.
- The ring $A^*(\mathcal{C}_g)$ is believed to be very large. One therefore concentrates on a subring of geometrically important classes, the so-called *tautological ring*, $R^*(\mathcal{C}_g)$. We shall define $R^*(\mathcal{C}_g)$ below.

Definition

Let $\pi : \mathcal{C}_g \rightarrow \mathcal{M}_g$ be the morphism that forgets the marked point, i.e. $\pi([C, p]) = [C]$. Let ω_π be its relative dualizing sheaf, define $K = c_1(\omega_\pi) \in A^1(\mathcal{C}_g)$ and let $\kappa_i \in A^i(\mathcal{C}_g)$ be the pullback of the class $\pi_*(K^{i+1}) \in A^i(\mathcal{M}_g)$. We now define $R^*(\mathcal{C}_g)$ to be the $\mathbb{Q}$-subalgebra of $A^*(\mathcal{C}_g)$ generated by the κ-classes and K.

Definition

The tautological ring of $\mathcal{M}_g$, $R^*(\mathcal{M}_g)$ is the $\mathbb{Q}$-subalgebra of $A^*(\mathcal{M}_g)$ generated by the classes $\pi_*(K^{i+1})$.

Theorem 1

[Faber, [1], Looijenga, [3]] $R^{g-1}(\mathcal{C}_g)$ is one-dimensional, generated by K^{g-1}, and $R^i(\mathcal{C}_g)$ vanishes for $i \geq g$. Similarly, $R^{g-2}(\mathcal{M}_g)$ is generated by $\pi_*(K^{g-1})$ and $R^i(\mathcal{M}_g)$ vanishes if $i \geq g-1$.

Thus, if one picks an element $a \in R^i(\mathcal{C}_g)$ and an element $b \in R^{g-1-i}(\mathcal{C}_g)$, then the element $a \cdot b$ will be a rational multiple of K^{g-1}. We order the κ-classes lexicographically and extend the order to $R^*(\mathcal{C}_g)$ by saying that $K^i\kappa_I > K^j\kappa_J$ if $i > j$ or if $i = j$ and $\kappa_I > \kappa_J$. We define $r^i_{k,l}$ as the number satisfying $m^i_k m^{g-1-i}_l = r^i_{k,l} K^{g-1}$, where m^i_k is the kth monomial of degree i and m^{g-1-i}_l is the lth monomial of degree l.

The numbers $r^i_{k,l}$ define a matrix, Q_i, of dimensions $\left(\sum_{j=0}^{i} p(j)\right) \times \left(\sum_{j=0}^{g-1-i} p(j)\right)$, where p is the partition function. Since a relation between the monomials would give rise to a relation between the rows we have that the rank of Q_i is a lower bound for the dimension of $R^i(\mathcal{C}_g)$.

Similarly, one may order the monomials of $R^*(\mathcal{M}_g)$ and define numbers $r^i_{k,l}$ by $m^i_k m^{g-2-i}_l = r^i_{k,l}$ to obtain matrices P_i. These matrices have been studied by Liu and Xu, [2], and they have developed efficient methods to compute the matrices P_i. It is therefore desireable to obtain a relation between the matrices Q_i and the matrices P_i.

Definition

For $j > 0$, let P^j_i be the submatrix of P_i consisting of the rows of P_i which corresponds to monomials which contains at least one factor $\pi_*(K^{j+1})$. Further, define $P^0_i = (2g-2)P_i$ and let P^{-1}_i be the zero matrix of dimensions $p(i+1) \times p(g-2-i)$.

Theorem 2

If $i \geq 1$, then

$$Q_i = \begin{pmatrix} P^{-1}_{i-1} & P^0_i & P^1_{i+1} & \cdots & P^{g-2-i}_{g-2} \\ P^0_{i-1} & P^1_{i+1} & \cdots & \cdots & \vdots \\ P^1_{i-1} & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ P^{i-1}_{i-1} & \cdots & \cdots & \cdots & P^{g-2}_{g-2} \end{pmatrix},$$

and the rank of Q_0 is 1.

Theorem 2 has been used to construct a Maple program which has computed the ranks of the matrices Q_i for all genera between 2 and 27, thus giving lower bounds for the dimensions of $R^i(\mathcal{C}_g)$.

A theorem of Faber, [1], gives a family of relations in the tautological ring of $\mathcal{M}_{g,2g-1}$. These relations can be pushed down to $R^*(\mathcal{C}_g)$. Using this method one may hope to find enough relations to show that the lower bound provided by the rank of Q_i is in fact the dimension of $R^i(\mathcal{C}_g)$. This has been done for $2 \leq g \leq 9$. Since the computations are quite hard, no higher genera has been attempted. However, the computations that has been performed show that:

Theorem 3

The tautological ring of $\mathcal{C}_g$ is Gorenstein for $2 \leq g \leq 9$.

References

[1] Faber, C., *A Conjectural Description of the Tautological Ring of the Moduli Space of Curves*, Moduli of Curves and Abelian Varieties: The Dutch Intercity Seminar on Moduli, Aspects Math. E33, Germany, 1999, 109–129.

[2] Liu, K., Xu, H., *Computing Top Intersections in the Tautological Ring of $\mathcal{M}_g$*, 2010, arXiv:1001.4498v1.

[3] Looijenga, E., *On the Tautological Ring of $\mathcal{M}_g$*, Invent. math. 121, 1995, 411-419.