Relative $\mathcal{D}$-modules in char $p > 0$

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Relatives $\mathcal{D}$-modules

LET $X \to S$ be a smooth morphism of schemes over a field k, let $U \subset X$ be an open with relative étale coordinates $x_1, \dots, x_n$, and let $\delta_{x_i,(n)} \in End_{\mathcal{O}_S}(\mathcal{O}_X)$ denote the differential operator defined by

$$\delta_{x_i,(n)}(x_j^m) = \delta_{ij}\binom{m}{n}x_j^{m-n},$$

then the module of *relative differential operators* $\mathcal{D}_{X/S}$ is a subset of $End_{\mathcal{O}_S}(\mathcal{O}_X)$ locally generated by $\delta_{x_i,(n)}$ over $\mathcal{O}_U$. Note that if $char\, k = 0$, then $\delta_{x_i,(n)} = \delta_{x_i,(1)}/n!$ hence $\mathcal{D}_{X/S}$ is generated by $Der_{\mathcal{O}_S}(\mathcal{O}_X)$ and $\mathcal{O}_X$. On the other side, if $char\, k = p > 0$ this is no longer true, as $p!$ is no longer invertible. Note also that $D_{X/S}$ acts naturally on $\mathcal{O}_X$, we will call this action the *trivial* action. Let M be a $\mathcal{O}_X$-module, then M endowed with an $\mathcal{O}_X$-linear action

$$\nabla : \mathcal{D}_{X/S} \to End_{\mathcal{O}_S}(M)$$

is called a *(left)* $\mathcal{D}_{X/S}$*-module* if the action ∇ extends the $\mathcal{O}_X$-action giving the module structure (for example, the zero action on M gives not raise to a $\mathcal{D}_{X/S}$-module). A $\mathcal{O}_X$-coherent $\mathcal{D}_{X/S}$-module is automatically locally free.

LET now k be an algebraically closed field of positive characteristic. If $S = Spec(k)$, then the category $\mathcal{D}_{\mathcal{O}coh}(X)$ of $\mathcal{O}_X$-coherent $\mathcal{D}_{X/k}$-modules toghether with morphisms commuting with the $\mathcal{D}_{X/k}$ action, is a k-linear *tannakian category*; taking the fiber in any rational point $x \in X(k)$ gives a neutral fiber functor $\omega_x : \mathcal{D}_{\mathcal{O}coh}(X) \to Vect_k$ into the category of finite k-vector spaces. For $M \in \mathcal{D}_{\mathcal{O}coh}(X)$ let $\langle M\rangle_\otimes$ be the smallest full tannakian subcategory of $\mathcal{D}_{\mathcal{O}coh}(X)$ containing M (hence containing $M, M^\vee$, closed for finite tensor products, finite sums and subquotients), then the Tannakian formalism ensure the existence of an affine k-group scheme $\pi_1(\langle M\rangle_\otimes, \omega_x)$ associated to $\langle M\rangle_\otimes$, called the *monodromy group* of M. An object (M, ∇) in $\mathcal{D}_{\mathcal{O}coh}(X)$ is called *trivial* if it is isomorphic to $\oplus\mathcal{O}_X$ where $\mathcal{D}_{X/S}$ acts trivially on every $\mathcal{O}_X$.

My research

FROM now on, let X, S be two smooth connected k-varieties and $X \to S$ a smooth morphism with geometrically connected fibres, let (M, ∇) be an $\mathcal{O}_X$-coherent $\mathcal{D}_{X/S}$-module. Following the idea of Grothendieck-Katz's p-curvature conjecture we would like to study the behavior of (M, ∇) on the generic fibre knowing its behavior on the closed ones. In particular we would like to know when it is *isotrivial*, i.e. there exist a finite étale cover that trivialize it. More precisely, the following theorem was proved by Y. André

Theorem 1. [An] ("Equicharacteristic 0 p-curvature")
Let $char k = 0$, then (M, ∇) is isotrivial on the generic fibre iff it is isotrivial on every closed point of an open of S.

ONE may ask then if the same holds also in equicharacteristic $p > 0$. This is not true in this generality but the following theorem holds:

Theorem 2. [EL] ("Equicharacteristic $p > 0$ p-curvature")
Let $char\, k = p > 0$, assume that $X \to S$ is proper *then if (M, ∇) has finite monodromy group of order prime to p on every closed point of a dense subset $S' \subset S(k)$, then if η indicates the generic point of S,*

i) there exist a finite Galois ètale covering $\pi_{\bar\eta} : Y_{\bar\eta} \to X_{\bar\eta} = X \times_S \bar\eta$ of order prime to p such that $\pi^*_{\bar\eta}(M_{\bar\eta}, \nabla)$ is a direct sum of line bundles

ii) if k is uncountable then $(M_{\bar\eta}, \nabla)$ trivializes on a finite ètale cover of order prime to p (factoring as a Kummer+Galois cover).

The natural question is then:

Q. *Does the Theorem 2 hold without the projective assumption on $X \to S$?*

Note that from a smooth case it is quite easy to reduce to a quasi-projective one.

References

[An] Y. André: Sur la conjecture des p-courbures de Grothendieck-Katz et un problème de Dwork, Geometric aspects of Dwork theory, Vol. I,II, 55-112, Walter de Gruyter GmbH & Co. KG, Berlin, 2004

[EL] H. Esnault, A. Langer: On a positive equicharacteristic variant of the p-curvature conjecture (2011,preprint)