Problems for GAeL XX; Samuel Grushevsky.

The problems go beyond the material covered in this course. The number of stars is correlated with relative difficulty. ** is something I don't know how to do without invoking heavy machinery not covered in the course, *** nobody (to the best of my knowledge) knows how to do — if you have any ideas for ** or ***, I would be very interested.

Problem 1. Prove that if A is an abelian variety (a complex projective variety with a group structure), its universal cover is $\mathbb{C}^g$.

Problem 2. Prove that θ is quasi-periodic in z, i.e. relate $\theta(z+m)$ to $\theta(z)$ for $m \in \mathbb{Z}^g$.

Problem 3. In dimension 1, prove that theta constants are modular forms (hint: Poisson summation formula for the Fourier series).

Problem 4. Prove honestly that $\wedge d\tau_{ij}$ is a modular form of weight $g+1$.

Problem 5 (*)**.** Is the derivative of a modular form a modular form? In genus 1, can you write down any polynomial of a theta constant and its derivatives that is a modular form?

Problem 6. Describe the moduli space $\mathcal{A}_1$ as explicitly as you can. Use this description to compactify it, and to compute $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{A}_1)$ and $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{A}}_1)$.

Problem 7 (*)**.** Do the same for $\mathcal{A}_2$. How would your answer differ for $\mathcal{M}_2$? What different compactifications can you think of, and how are they related? How are their Picard groups related?

Problem 8. Determine the dimension of $H^0(\overline{\mathcal{A}}_1, 2L)$, and use this to deduce the Riemann identity

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^4 (\tau) = \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}^4 (\tau) + \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}^4 (\tau).$$

Problem 9 (* (hard))**.** Determine the ring of modular forms of genus 1, i.e. compute $\oplus_{n\geq 0} H^0(\overline{\mathcal{A}}_1, nL)$.

Problem 10 (*)**.** For $g=2$, use theta constants to define two modular forms (modular for the entire $Sp(4,\mathbb{Z})$) satisfying an algebraic relation.

Problem 11 (*)**.** Define an $\mathrm{Sp}(2g,\mathbb{R})$ action on the Siegel upper half-space $\mathcal{H}_g$, and determine the stabilizer of some point $\tau \in \mathcal{H}_g$. Use this to define a metric on $\mathcal{H}_g$ invariant under the $\mathrm{Sp}(2g,\mathbb{R})$ action, the so-called Siegel or Bergman metric.

Problem 12 (*)**.** Compute the volume of $\mathcal{A}_1$ and of $\mathcal{A}_2$ in the metric defined in the previous problem.

Problem 13. Computing the volume of $\mathcal{A}_3$ in this metric explicitly is hard. Thus express the volume of $\mathcal{A}_g$, for any g, as an intersection number (hint: what is the class of the (1,1)-form giving the metric?)

Problem 14 (***)**.** What can you say about the curvature of the locus of Jacobians $\mathcal{J}_g \subset \mathcal{A}_g$ with respect to the metric defined above?

Problem 15. Let $F_1(\tau) := \prod\limits_{m \text{ even}} \theta_m(\tau)$, so that $\Theta_{\text{null}} = \{F_1(\tau) = 0\} \subset \mathcal{A}_g$. Use the geometry of the $g = 3$ case to compute the class of the closure of the hyperelliptic locus in $\overline{\mathcal{M}}_3$ (watch out for the divisors δ_i).

Problem 16 (*)**.** Compute the class of the theta-null divisor in $\overline{\mathcal{A}}_g$ and $\overline{\mathcal{M}}_g$ for any g (in the lecture we only discussed $\mathcal{A}_g$ and did not compute the boundary coefficient for it).

Problem 17 (*)**.** Let

$$F_2(\tau) := \prod_{m \text{ even}} \theta_m^8(\tau) \cdot \sum_{n \text{ even}} \theta_n^{-8}(\tau)$$

What is the geometric significance of the locus $\{F_1(\tau) = F_2(\tau) = 0\}$? Use this (and the solution to the Schottky problem in genus 4) to compute the class of the hyperelliptic locus in $\mathcal{M}_4$. (hint: every genus 4 hyperelliptic curve has 10 theta-nulls — interpret and prove this in terms of g^r_d's).

Problem 18 (**)**.** Use the above to compute the class of the closure of the hyperelliptic locus in $H^4(\overline{\mathcal{M}}_4)$.

Problem 19 (**)**.** Let

$$F_3(\tau) := \prod_{m \text{ even}} \theta_m^8(\tau) \cdot \sum_{n_1 \neq n_2 \text{ even}} \theta_{n_1}^{-8}(\tau)\theta_{n_2}^{-8}(\tau)$$

Is the locus $\{F_1(\tau) = F_2(\tau) = F_3(\tau) = 0\} \subset \mathcal{A}_g$ irreducible? What about its intersection with $\mathcal{J}_g$?

Problem 20 (***)**.** Can the above be used to characterize hyperelliptic curves, i.e. can you write down a collection of modular forms defining the hyperelliptic locus in $\mathcal{J}_g$? Is the hyperelliptic locus a complete intersection? If not, what is the minimal number of defining equations?

Problem 21 (***)**.** Compute $\dim H^0(\overline{\mathcal{A}_g}, N_0')$.

Problem 22 (***)**.** Compute the codimension of the locus $N_1 \subset \mathcal{A}_g$.