Prolems for GAEL XX

1 Lecture 1

1. Let $\mathbb{Z}/2$ acts by -1 on the affine line $\mathbb{A}^1$. Show that the quotient sheaf (e.g. for the fppf topology) $\mathbb{A}^1/(\mathbb{Z}/2)$ is not representable by a scheme. The same question for $\mathbb{G}_m$ acting by multiplication on $\mathbb{A}^1$.

2. Let R be a finitely generated associative k-algebra. Define M_R^s to be the (set valued) moduli functor classifying simple and finite dimensional R-modules. Give a precise definition of the corresponding moduli functor. Is it a sheaf (for the fppf or etale topology) ? Is it representable by a scheme ? Is its associated sheaf representable by a scheme ?

3. Let S be a smooth algebraic variety, $Y, Z \subset S$ be two smooth closed sub-varieties, and X their intersection. Let A be a commutative k-algebra of finite type and I an ideal in A with $I^2 = 0$. Construct, for any morphism of k-schemes $x : Spec\, A/I \to X$, en element

$$o(x) \in Coker\left((T_{Y,x} \oplus T_{Z,x}) \otimes_{A/I} I \longrightarrow T_{S,x} \otimes_{A/I} I\right),$$

in such a way that $o(x) = 0$ if and only if x extends to a morphism $Spec\, A \to X$ (here $T_{Y,x}$, $T_{Z,x}$ and $T_{S,x}$ are the pull-backs of the tangent sheaves on $Spec\, A/I$).

4. Let R be a finitely generated associative k-algebra and V a finite dimensional R-module. Define the moduli functor of subobjects of V and prove that it is representable by a projective scheme. Use this to prove the following statement: if V is simple and if $Ext^1_R(V,V) \neq 0$ and $Ext^2_R(V,V) = 0$, then there is an infinite number of non-isomorphic simple R-modules.

2 Lecture 2

1. Compute the cotangent complexes of $A = k[t]/t^n$, as well as the morphisms between them induced by the natural projections $k[t]/t^{n+1} \to k[t]/t^n$.

2. Compute the derived tensor product $k \otimes^{\mathbb{L}}_{k[t]/t^n} k$. Show that for any commutative dg-algebra A with an augmentation $A \to k$, the derived tensor product $k \otimes^{\mathbb{L}}_A k$ can be written (up to a quasi-isomorphism) as $Sym_k(L[1])$ for a certain graded k-module L. Can you identify L in terms of the cotangent complex $\mathbb{L}_A$ of A ?

3. Let A be a commutative k-algebra of finite type and I be an ideal. We assume that locally on $Spec\, A$, I is generated by a regular sequence ($Spec\, A/I \hookrightarrow Spec\, A$ is l.c.i.). Compute the derived tensor product $A/I \otimes^{\mathbb{L}}_A A/I$ and show that it is quasi-isomorphic to $Sym_A(I/I^2[1])$.

4. Let A be commutative k-algebra of finite type which is a local complete intersection. Show that for any commutative dg-algebra B with $\pi_0(B) = A$, the natural projection $B \longrightarrow \pi_0(B) = A$ has a section in the homotopy category. Show that this property characterises local complete intersections. Deduce that for a general commutative dg-algebra B the projection $B \to \pi_0(B)$ has no sections up to homotopy.

5. Show that flat, smooth and etale morphisms of commutative dg-algebras are stable by compositions and derived base changes. Prove that smooth and etale morphisms of commutative dg-algebras are local for the etale topology.

6. Show that if $A \to B$ is an etale morphism of commutative dg-algebras, then for any commutative A-algebra C, the mapping space $Map_{A-cgda}(B, C)$ is homotopically discrete (i.e. all π_i vanish for $i > 0$).

7. Show that a commutative dg-algebra A is finitely presented if and only if it satisfies the following two conditions:

   (a) $\pi_0(A)$ is finitely presented as a commutative k-algebra.

   (b) the cotangent complex $\mathbb{L}_A$ is finitely presented as a A-dg-module.

8. Let A be a quasi-smooth ommutative dg-algebra (i.e. it is finitely presented and its cotangent complex $\mathbb{L}_A$ has amplitude in [-1,0]). Show that, locally for the Zariski topology, there exists a commutative smooth k-algebra B, a projective B-module of finite type M and a morphism of B-modules $s : M \to B$, such that A is quasi-isomorphic to the Koszul dg-algebra $Sym_B(M[1])$ whose differential is induced by s. Geometrically: any quasi-smooth derived scheme is locally the derived locus of zeros of a section of vector bundle on a smooth scheme.

9. Let $A = Sym_k(k[n])$ for $n \leq 0$. Show that k is a coherent A-dg-module and is perfect if and only if n is even.

3 Lectures 3

1. Provide a complete proof that derived schemes are stable by homotopy pulll-backs inside derived stacks.

2. Construct an example of a derived scheme X with a smooth truncation $h^0(X)$, but such that the inclusion $h^0(X) \hookrightarrow X$ does not possess a retraction. Show that such examples can not exist when $h^0(X)$ is of dimension less than 1.

3. Let $f : X \longrightarrow X'$ be a morphism between derived schemes of finite type over k. Suppose that

   (a) the induced map $X(\overline{k}) \to X'(\overline{k})$ is a bijective map of sets

   (b) for all $x \in X(\overline{k})$, the induced morphism on tangent complexes

   $$T_{X,x} \longrightarrow T_{X',f(x)}$$

   is a quasi-isomorphism.

   Prove that f is an equivalence of derived schemes.

4. Let

   $$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

   be a commutative triangle of derived schemes of finite type over k. Show that f is etale (resp. smooth) if and only if for all geometric point $s \in S(\overline{k})$ the induced morphism on the derived fibers

   $$f_s : p^{-1}(\{s\}) \longrightarrow q^{-1}(\{s\})$$

   is an etale (resp. smooth).

5. Let $S = Spec\, A$ be a derived affine scheme such that $\pi_*(A)$ is finite dimensional over k (i.e. A is an Artin dg-algebra). Show that for all derived Artin stack X, the derived mapping stack $\mathbb{R}\underline{Map}(S, X)$ is representable by a derived Artin stack.

6. Let X be a derived Artin stack. Show that $h^0(X)$ carries canonical quasi-coherent sheaves $\pi_i(\mathcal{O}_X)$. Provide examples of non-equivalent such X giving rise to the same $h^0(X)$ and the same sheaves $\pi_i(\mathcal{O}_X)$.

7. Let X be a derived Artin stack with $\pi_i(\mathcal{O}_X) = 0$ for i big enough. Show that the inclusion $i : h^0(X) \hookrightarrow X$ induces an isomorphism on the Grothendieck groups of coherent modules
$$i_* : G_0(h^0(X)) \simeq G_0(X).$$

8. Let $f : X \longrightarrow Y$ be a morphism of derived Artin stack. Show that if f is quasi-smooth then the derived pull-back
$$f^* : D_{qcoh}(Y) \longrightarrow D_{qcoh}(X)$$
preserves coherent modules and thus induces a pull-back
$$f^! : G_0(Y) \longrightarrow G_0(X).$$
Using the previous exercise, this induces a pull-back
$$f^{!!} : G_0(h^0(Y)) \longrightarrow G_0(h^0(X)).$$
Is this induced by the derived pull-back of coherent sheaves on $h^0(X) \longrightarrow h^0(Y)$? Suppose that $X = h^0(X)$, what is then $f^{!!}(\mathcal{O}_X)$?

4 Lecture 4

1. Let Z be a smooth Artin stack, V a vector bundle on X and $s : X \to V$ a section. Let X be the derived Artin stack of zeros of s. Show that the virtual class of X is $C_{top}(V)$, the top Chern class of V.

2. Let G be an algebraic group with Lie algebra $\mathfrak{g}$ and K be a finite CW complex. Show that $\mathbb{R}\underline{Map}(K,G)$ decomposes, as a derived scheme, as
$$\mathbb{R}\underline{Map}(K,G) \simeq G^{\pi_0(K)} \times Spec\ (Sym_k(V \otimes \mathfrak{g}^*))\,,$$
where $V = \bigoplus_{i>0} H_i(K,k)[i]$ is the reduced homology of K.

3. Let G be an algebraic group with Lie algebra $\mathfrak{g}$. Show that we have
$$\mathbb{R}Loc_G(S^n) \simeq [Spec\, A/G],$$
where $A = Sym_k(\mathfrak{g}^*[n-1])$ for $n > 1$. What is the formula for $n = 0$ and $n = 1$?

4. Let G be a reductive group. Compute the virtual class of $\mathbb{R}Loc_G(S^2)$. Show that it vanishes for $G = Gl_n$.

5. Let S be a smooth and projective surface, and let $\mathbb{R}Pic(S) := \mathbb{R}\underline{Map}(S, B\mathbb{G}_m)$ be the derived Picard stack of S. Show that $\mathbb{R}Pic(S)$ is quasi-smooth and has a trivial virtual class. Deduce that $\mathbb{R}Vect(S)$, the derived moduli stack of vector bundles on S also has a trivial virtual class.

6. Let X be a K3 surface and C a smooth proper curve C of genus $g \geq 2$. Construct a morphism of derived Artin stacks

$$\Theta : \mathbb{R}\underline{Map}(C, X) \longrightarrow \mathbb{R}Pic(X),$$

sending a map $f : C \longrightarrow X$ to the determinant of the coherent complex $f_*(\mathcal{O}_C)$. Show that this map is quasi-smooth and deduce that the virtual class of $\mathbb{R}\underline{Map}(C, X)$ vanishes.

7. We consider the triangulated category $T := D_{qcoh}(\mathbb{R}Loc_G(S^2))$. Construct a monoidal structure on T (consider a 3-sphere with three holes M and $\mathbb{R}Loc_G(M)$ to create a correspondence). Prove that it is compatible with the standard t-structure on T, and that the corresponding heart is $Rep(G)$, the usual tensor category of linear representations of G. Is T equivalent, as a tensor triangulated category, to $D(Rep(G))$, the derived category of linear representations of G ?

8. Let X be a smooth and projective complex variety, and $\beta \in H^2(X, \mathbb{Z})$ a fixed class. Define a derived stack of stable maps $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta)$ (genus g, n maked points and class β). Explain how to use it to define functors

$$\Theta_{g,n,\beta} : D^b_{coh}(\overline{\mathcal{M}}_{g,n+1}) \times D^b_{coh}(X^n) \longrightarrow D^b_{coh}(X).$$

Can you see a relation between the $\theta_{g,n,\beta}$'s and the Gromov-Witten theory of X ?

9. Compute the de Rham complex of BG for G a reductive group. Show that symplectic forms shifted by 2 on BG are in one-to-one correspondence with non-degenerate bilinear G-invariants forms on $\mathfrak{g} := Lie(G)$.

10. Let M be an oriented n-dimensional topological manifold with boundary $\partial M = N$. Show that the restriction map

$$r : \mathbb{R}Loc_G(M) \longrightarrow \mathbb{R}Loc_G(N)$$

is such that $r^*(\omega) = 0$, where ω is the canonical $(2 - n + 1)$-shifted symplectic structure on $\mathbb{R}Loc_G(N)$. Prove that the derived fibers of r carry natural $(2 - n)$-shifted symplectic structures.