Real and Convex Algebraic Geometry

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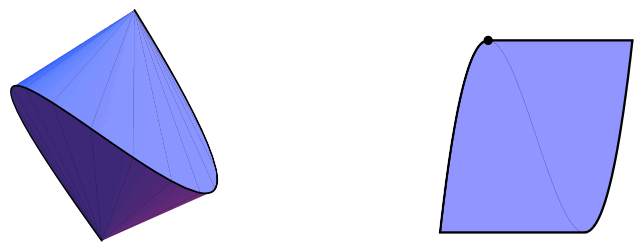
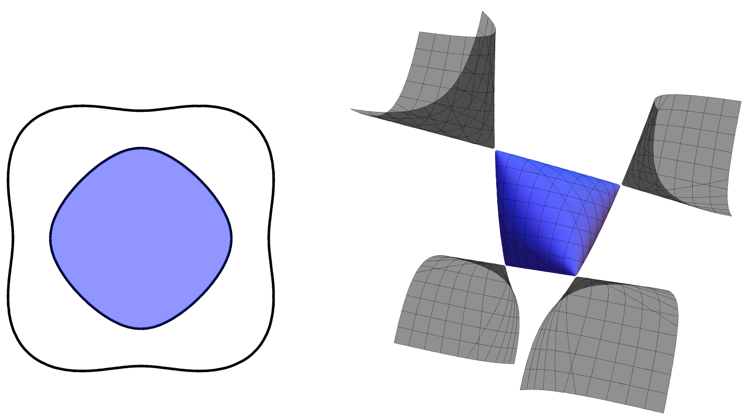
I am interested real algebraic geometry and connections to convexity and optimization. Central objects in this study are affine linear slices of the cone of symmetric, positive semidefinite matrices

$$\{\mathbf{x} \in \mathbb{R}^n \; : \; A_0 + \sum_i x_i A_i \succeq 0\},$$

called **spectrahedra**, and their linear projections.



Their boundaries are **hyperbolic hypersurfaces** with definite determinantal representations $\det(A_0 + \sum_i x_i A_i)$.



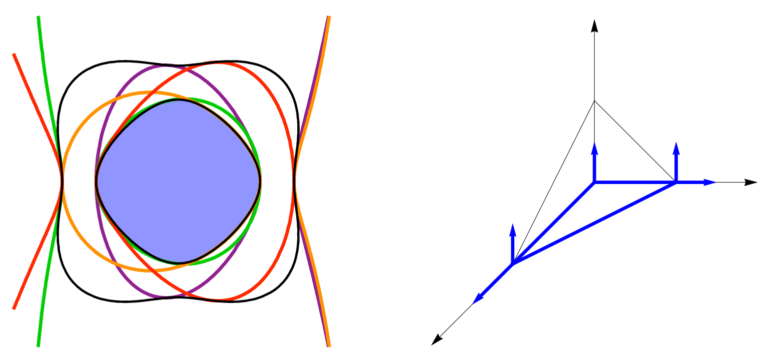
One of the big challenges in this field is to understand (and construct) such determinantal representations of these hypersurfaces.

Contact Curves and Determinantal Representations

One can construct symmetric determinantal representations of curves in $\mathbb{P}^2$ via their *contact curves*. If $f = \det(A(\mathbf{x}))$ where $A(\mathbf{x}) = \sum_i x_i A_i$, for matrices $A_i \in \mathbb{C}^{d\times d}_{sym}$, then the degree $d-1$ curve

$$g = \det \begin{pmatrix} A(\mathbf{x}) & \mathbf{v} \\ \mathbf{v}^T & 0 \end{pmatrix}$$

meets f in $\frac{d(d-1)}{2}$ points of multiplicity 2 for any $\mathbf{v} \in \mathbb{C}^d \backslash 0$. Conversely, from such a curve, one can reconstruct the determinantal representation $A(\mathbf{x})$ [1].



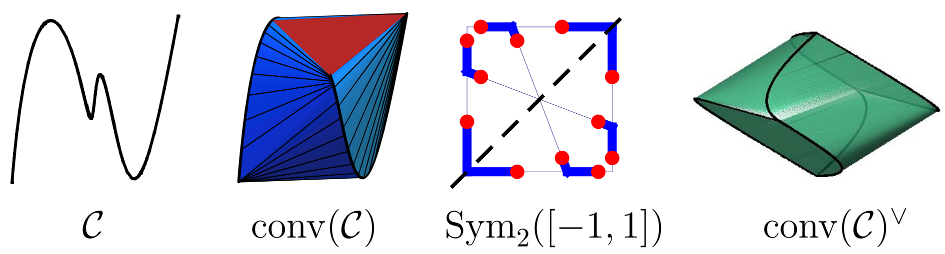
Tangency and Convex Hulls

Q: What is the algebraic boundary of the convex hull of a curve? its degree? its irreducible components?

Here's my favorite example:

$$\mathcal{C} = \{(t,\; 4t^3 - 3t,\; 16t^5 - 20t^3 + 5t) \; : \; t \in [-1,1]\} \subset \mathbb{R}^3.$$

The Zariski closure of the boundary of its convex hull, $\mathrm{conv}(\mathcal{C})$, is cut out by a union of irreducible hypersurfaces of degrees 1, 1, 4, 4, and 7. One can also ask for set of vertices $\{a_1, a_2\} \in \mathrm{Sym}_2([-1,1])$ of edges on $\mathrm{conv}(\mathcal{C})$:



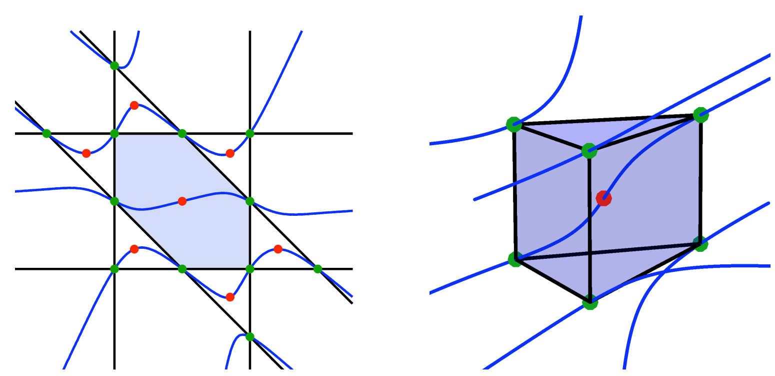
The algebraic boundary of the dual convex body, $\mathrm{conv}(\mathcal{C})^\vee$, is the union of two hyperplanes and a degree-7 component given by the discriminant of $\mathbb{R}\{1, t, t^3, t^5\}$.

Central Curves in Optimization

Another interesting way algebraic geometry arises in optimization is in the study of central curves of linear programs. Given a linear space $\mathcal{L}$, we can define

$$\mathcal{L}^{-1} = \overline{\left\{ \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n} \right) \; : \; \mathbf{x} \in \mathcal{L} \cap (\mathbb{C}^*)^n \right\}}.$$

The **central curve** associated to $\mathcal{L}$ and two additional vectors $\mathbf{c}, \mathbf{v} \in \mathbb{R}^n$ is given by $(\mathcal{L} + \mathrm{span}(\mathbf{c}))^{-1} \cap (\mathcal{L}^\perp + \mathbf{v})$.



One can write the degree and genus of these curves in terms of matroid invariants and obtain bounds on their curvature [2].

References

[1] Dixon: *Note on the reduction of a ternary quantic to a symmetrical determinant*, Cambr. Proc. **11** (1902) 350–351.

[2] de Loera, Sturmfels, and Vinzant, *The central curve in linear programming*, `arXiv:1012.3978`.