Models for Threefolds Fibred by K3 Surfaces of Degree Two

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This talk is based upon my current work on the explicit construction of models for threefolds fibred by K3 surfaces of degree two. Its contents may be found in more detail in the preprint [Tho11a] and in my doctoral thesis [Tho11b], a copy of which is currently available on my website:

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We begin by defining our objects of study:

Definition 1 *A* K3 surface of degree two *is a nonsingular projective surface X satisfying $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$, along with an ample invertible sheaf $\mathcal{L}$ on X that has self-intersection number $\mathcal{L}.\mathcal{L} = 2$.*

A Riemann-Roch calculation shows that such surfaces can be seen as double covers of the projective plane $\mathbb{P}^2$ ramified over smooth sextic curves.

We are interested in studying threefolds that admit fibrations by such surfaces. Formally we define:

Definition 2 *Let S be a nonsingular complex curve. A* threefold fibred by K3 surfaces of degree two *over S is a triple $(X, \pi, \mathcal{L})$ satisfying:*

- *X is a nonsingular 3-dimensional complex variety;*
- *$\pi \colon X \to S$ is a projective, flat, surjective morphism with connected fibres, whose general fibres are K3 surfaces;*
- *$\mathcal{L}$ is an invertible sheaf on X which induces an ample invertible sheaf $\mathcal{L}_s$ on a general fibre X_s, that has self-intersection number $\mathcal{L}_s.\mathcal{L}_s = 2$.*

In order to study such threefolds, we would like to find explicit birational models for them. Using the fact that a K3 surface of degree two can be seen as a double cover of $\mathbb{P}^2$ ramified over a sextic, we first attempt to construct such models as double covers of $\mathbb{P}^2$-bundles. We obtain:

Theorem 3 [Tho11b, Theorem 1.3.3]. *Let* $(X, \pi, \mathcal{L})$ *be a threefold fibred by K3 surfaces of degree two. There exists a threefold* W *admitting a projective fibration* $p\colon W \to S$ *that is birational to* X *over* S *and that can be described explicitly as a double cover of a* $\mathbb{P}^2$*-bundle on* S.

Unfortunately, this model turns out not to be very good. To see why, we need to study the degenerations of K3 surfaces of degree two.

To see how a K3 surface of degree two may degenerate, we begin by relaxing the assumptions that we made in the definition of these surfaces. Specifically, we relax the condition that the invertible sheaf $\mathcal{L}$ must be ample, and instead assume that it is merely nef and big. In this case, we see that a second type of surface can arise: one that admits a morphism to a double cover of a quadric cone, ramified over a sextic curve and the vertex of the cone. Such surfaces cannot be seen as double covers of $\mathbb{P}^2$, so any fibres in $\pi\colon X \to S$ having this form are destroyed and replaced by the birational map $X - \to W$.

In fact, the presence of such "unigonal" fibres forms the only obstruction to the usefulness of W. We have:

Theorem 4 [Tho10, Theorem 4.1]. *Let* $\pi\colon X \to \Delta := \{z \in \mathbb{C} : 0 \leq |z| < 1\}$ *be a semistable* (*i.e. the central fibre is reduced and has normal crossings*) *degeneration of K3 surfaces with* $\omega_X \cong \mathcal{O}_X$ *and let* H *be a divisor on* X *that is effective, nef and flat over* Δ*, that induces an ample divisor* H_s *with* $H_s.H_s = 2$ *on a general fibre. Then* H *defines a morphism that maps the central fibre to either:*

- *A double cover of* $\mathbb{P}^2$ *ramified over a sextic curve; or*
- *A double cover of a quadric cone, ramified over a sextic curve and the vertex of the cone.*

Note that these surfaces may be singular (*even non-normal*).

Following an idea of Catanese and Pignatelli [CP06], we use this to refine our construction. Instead of a $\mathbb{P}^2$-bundle, we construct a bundle of rational surfaces on S, which we allow to degenerate to quadric cones. Taking a double cover of this bundle, we will obtain a new model $\pi^c\colon X^c \to S$. As the fibres in this model are no longer forced to admit a morphism to $\mathbb{P}^2$, this model takes much better account of the unigonal fibres.

In order to construct this model, we need to alter some of the assumptions on $(X, \pi, \mathcal{L})$ that we made originally. This will allow us to use the explicit description of the fibres obtained from Theorem 4. We assume:

- The polarisation $\mathcal{L}$ is locally flat, i.e. for all $s \in S$ there exists an open set $U_s \ni s$ and a section in $\Gamma(\pi^{-1}(U_s), \mathcal{L})$ that defines an effective and flat divisor over U_s;
- X is allowed to have mild (Gorenstein terminal) singularities;
- There exists an analytic resolution $f\colon Y \to X$ of X such that Y is semistable and the exceptional locus of f has codimension 2 in Y.

Under these new assumptions, the model that we would like to explicitly construct is the *relative log canonical model* of $(X, \pi, \mathcal{L})$, defined as:

$$X^c := \mathbf{Proj}_S\big(\bigoplus_{n \geq 0} \pi_*(\omega_X^{\otimes n} \otimes \mathcal{L}^{\otimes n})\big).$$

This model has been widely studied in relation to the minimal model program. In particular, it has the following desirable properties:

- There exists a birational map $\phi\colon X - \to X^c$ over S;
- The exceptional set of ϕ^{-1} has codimension 2 in X^c (so ϕ cannot "destroy and replace" any fibres);
- X^c has only mild (canonical) singularities.

As mentioned before, this model can be explicitly constructed as a double cover of a rational surface bundle on the base curve S. Furthermore, this construction depends only upon a relatively small set of data on S:

Theorem 5 [Tho11a, Theorem 6.2] *Any threefold fibred by K3 surfaces of degree two uniquely determines a certain* 5*-tuple of data on S, from which its relative log canonical model can be explicitly reconstructed.*

Furthermore, (*under certain conditions*) *any such* 5*-tuple of data defines a threefold that arises as the relative log canonical model of some threefold fibred by K3 surfaces of degree two.*

References

[CP06] F. Catanese and R. Pignatelli, *Fibrations of low genus I*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 6, 1011–1049.

[Tho10] A. Thompson, *Degenerations of K3 surfaces of degree two*, Preprint, October 2010, `arXiv:1010.5906`.

[Tho11a] ———, *Explicit models for threefolds fibred by K3 surfaces of degree two*, Preprint, January 2011, `arXiv:1101.4763`.

[Tho11b] ———, *Models for threefolds fibred by K3 surfaces of degree two*, Ph.D. thesis, University of Oxford, 2011, to appear.