Invariant Hilbert Schemes

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In particular, $k[X]$ is a G-module.

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Some answers :

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• If G is a connected reductive group and X a smooth variety such that $k[X]$ is a multiplicity free G-module, then $k[X]$ as a G-module determines $k[X]$ as a G-algebra up to a G-isomorphism. (I. Losev, 2009)

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We can check that $X_0 \ncong X_1$ but $k[X_0] \cong k[X_1]$ as G-modules by flatness.

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• In 2003, Alexeev and Brion give a construction of the invariant Hilbert scheme for G a connected reductive group in "Moduli of affine schemes with reductive group action ".

• Finally, in 2010, Brion gives a construction of the invariant Hilbert scheme for G a reductive group (without connectedness assumption) in the survey paper "Invariant Hilbert schemes".

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We have the following set-theoritically description of the **invariant Hilbert scheme** :

$$\mathrm{Hilb}_h^G(W) = \left\{ X \subset W \;\middle|\; \begin{array}{l} X \text{ is a } G\text{-stable closed subscheme such that} \\ k[X] \cong \bigoplus_{M \in Irr(G)} h(M)M \text{ as } G\text{-module} \end{array} \right\}$$

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The invariant Hilbert scheme encodes all the differents structures of quotient G-algebra induced by a given structure of G-module.

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$$S \mapsto \left\{ \begin{array}{c|l} \mathcal{Z} \subset W \times S & \mathcal{Z} \ G\text{-stable closed subscheme,} \\ \pi \searrow \quad \downarrow & \pi \text{ flat morphism,} \\ S & \forall s \in S,\ k[\mathcal{Z}_s] \cong \bigoplus_{M \in Irr(G)} h(M)M \end{array} \right\}$$

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Alexeev and Brion have shown that this functor is represented by a quasi-projective scheme noted $\mathrm{Hilb}^G_h(W)$ and called **invariant Hilbert scheme**.

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It follows that all the fibers of ν over U have the same Hilbert function. We call such a fiber the **generic fiber** of ν.

Our fist tool to compute the invariant Hilbert scheme $\mathrm{Hilb}_h^G(W)$ is the existence of the morphism (when $h(V_0) = 1$) :

$$\gamma : \ \mathrm{Hilb}_h^G(W) \to W//G$$

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In particular, $H^O := \overline{\gamma^{-1}(U)}$ is an irreducible component of $\mathrm{Hilb}_h^G(W)$.

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where, in case you forgot, $H^O := \overline{\gamma^{-1}(U)}$ is an irreducible component of $\operatorname{Hilb}_h^G(W)$.

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Proposition (-)

To each $M \in Irr(G)$ we can associate a morphism

$$\delta_M : \mathrm{Hilb}_h^G(W) \to \mathrm{Gr}(h(M), F_M^*)$$

where $F_M \subset \mathrm{Hom}^G(M, k[W])$ is some finite dimensional vector space.

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$$T_X \mathrm{Hilb}_h^G(W) \cong \mathrm{Hom}_R^G(I/I^2, R)$$

where $R := k[W]/I$ and Hom_R^G stands for the space of R-linear, G-equivariant maps.

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We want to determine $\mathrm{Hilb}_h^G(W)$ in this situtation.

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And in all cases, we show that

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gives a resolution of singularities.

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- *If* $m \geq 2$, *then* $\mathrm{Hilb}_h^G(W) \cong Bl_X(W//G)$ *where* X *is the singular locus of* $W//G$.

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Unfortunately, if $\dim(V) > 2$, we have examples where $\mathrm{Hilb}_h^G(W)$ is not smooth anymore for $G = GL(V),\ SO(V),\ O(V)$.

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When $\mathrm{Hilb}_h^G(W)$ *is not smooth, does* $H^0 := \overline{\gamma^{-1}(U)}$ *give a resolution of singularities ?*

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and we want to know when it gives a resolution of singularities.