ULRICH BUNDLES AND VARIETIES OF WILD REPRESENTATION TYPE

Joan Pons
Universitat de Barcelona
Advisors: Dr.Enrique Arrondo, Dra.Rosa Maria Miró.

Setting

We work in the setting of ACM schemes:

Definition 0.1. A closed subscheme $X \subseteq \mathbb{P}^n$ is *Arithmetically Cohen-Macaulay (briefly, ACM)* if its homogeneous coordinate ring R_X is a Cohen-Macaulay ring or, equivalently, $\dim R_X = \mathrm{depth}\ R_X$.

We are interested in constructing large families of indecomposable coherent sheaves on ACM varieties such that they are cohomologically as "simple" as possible, i.e., they are ACM:

Definition 0.2. Let $X \subseteq \mathbb{P}^n$ be a closed ACM scheme. A coherent sheaf $\mathcal{E}$ on X is *Arithmetically Cohen-Macaulay* (ACM for short) if it is locally Cohen-Macaulay (i.e., $\mathrm{depth}\,\mathcal{E}_x = \dim \mathcal{O}_{X,x}$ for every point $x \in X$) and has no intermediate cohomology:

$$\mathrm{H}^i_*(X, \mathcal{E}) = 0 \qquad \text{for all } i = 1, \ldots, \dim X - 1.$$

The ACM varieties with a finite number of ACM indecomposable coherent sheaves have been classified:

Theorem 0.3 ((Buchweitz, Greuel, Schreyer, Eisenbud, Herzog))**.** *Let $X \subseteq \mathbb{P}^n$ be an ACM variety of finite representation type. Then X is either three or less reduced points on $\mathbb{P}^2$, a projective space, a smooth quadric hypersurface $X \subset \mathbb{P}^n$, a cubic scroll in $\mathbb{P}^4$, the Veronese surface in $\mathbb{P}^5$ or a rational normal curve.*

On the other extreme would lie varieties of "wild representation type":

Definition 0.4. An ACM variety $X \subseteq \mathbb{P}^n$ is of *wild representation type* if there exist l-dimensional families of non-isomorphic indecomposable ACM sheaves for arbitrary large l.

Among ACM sheaves, we are specially interested in those with a large number of twisted global sections. There is a bound for this number:

Theorem 0.5 (Ulrich, Casanellas-Hartshorne)**.** *Let $X \subseteq \mathbb{P}^n$ be an integral subscheme and $\mathcal{E}$ be an ACM sheaf on X of positive rank. Then the minimal number of generators $m(\mathcal{E})$ of the R_X-module $\mathrm{H}^0_*(\mathcal{E}) := \oplus_i \mathrm{H}^0(X, \mathcal{E}(i))$ is bounded by*

$$m(\mathcal{E}) \leq \deg(X)\,\mathrm{rk}(\mathcal{E}).$$

Definition 0.6. An ACM sheaf $\mathcal{E}$ of positive rank will be called an *Ulrich sheaf* if $m(\mathcal{E}) = \deg(X)\,\mathrm{rk}(\mathcal{E})$.

Why are Ulrich $\mathcal{O}_X$-sheaves meaningful? Because they give relevant information about the structure of X. For instance, in the algebraic case:

Theorem 0.7 ((Ulrich))**.** *If a Cohen-Macaulay ring R supports an Ulrich module M verifying $\mathrm{Ext}^i_R(M, R) = 0$ for $1 \leq i \leq \dim(R)$, then R is Gorenstein.*

Results

Our contribution regards varieties $X \cong Bl_Z\mathbb{P}^n$ which are a blow-up of a finite set of points Z on $\mathbb{P}^n$ with ample anticanonical divisor, i.e, they are Fano:

Definition 0.8. A *Fano variety* is defined to be a smooth n-variety X whose anticanonical divisor $-K_X$ is ample. Its degree is defined as $(-K_X)^n$. If $-K_X$ is very ample, X will be called a *strong Fano variety*.
A two-dimensional Fano variety (resp. strong Fano variety) is called a *del Pezzo surface* (resp. *strong del Pezzo surface*).

Del Pezzo surfaces have been classically classified:

Theorem 0.9. *Let X be a del Pezzo surface of degree d. Then $1 \leq d \leq 9$ and*

(i) If $d = 9$, then X is isomorphic to $\mathbb{P}^2_k$ (and $-K_{\mathbb{P}^2_k} = 3H_{\mathbb{P}^2_k}$ gives the usual Veronese embedding in $\mathbb{P}^9_k$).

(ii) If $d = 8$, then X is isomorphic to either $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or to a blow-up of $\mathbb{P}^2_k$ at one point.

(iii) If $7 \geq d \geq 1$, then X is isomorphic to a blow-up of $9 - d$ points in general position.

Theorem 0.10 ((Miró-Roig -P.))**.** *Let $X \cong Bl_Z\mathbb{P}^n \subseteq \mathbb{P}^N$ be a Fano blow-up embedded in $\mathbb{P}^N$ by the anticanonical divisor $-K_X$, $n \geq 3$ and let $r \geq n$. Then there exists a family of rank r simple (hence, indecomposable) ACM vector bundles of dimension $\sim r^2$. In particular, Fano blow-ups are varieties of wild representation type*

In the two dimensional case, i.e, for del Pezzo surfaces, we obtain a much stronger result:

Theorem 0.11 ((Miró-Roig -P.))**.** *Let X be a del Pezzo surface of degree d with anticanonical divisor $H := -K_X$. Then for any $r \geq 2$ there exists a family of dimension $r^2 + 1$ of simple initialized Ulrich vector bundles of rank r with Chern classes $c_1 = rH$ and $c_2 = \frac{dr^2+(2-d)r}{2}$. Moreover, they are μ-semistable with respect to the polarization $H = 3e_0 - \sum_{i=1}^{9-d} e_i$ and μ-stable with respect to $H_n := (n-3)e_0 + H$ for $n \gg 0$. In particular, del Pezzo surfaces are of wild representation type.*