Moduli spaces of rational tropical curves

Dennis Ochse
TU Kaiserslautern
Fachbereich Mathematik

Advisor: Prof. Andreas Gathmann

Introduction

MODULI SPACES of stable maps $\overline{M}_{0,n}(X,\beta)$ are well known in algebraic geometry. Their points are morphisms from a rational curve with n marked points into X such that the image has homology class β. These spaces have the structure of a Deligne-Mumford stack and in particular there is an intersection theory on these spaces which enables to do enumerative geometry. On the side of tropical geometry there are not even corresponding moduli spaces which have a tropical structure yet, except for a few special cases. One of these cases is the space $\mathcal{M}_{0,n}(\mathbb{R}^r,\Delta)$ which is a tropical fan (a weighted polyhedral fan which is balanced around codimension one cones). The points of this space correspond to tropical stable maps, i.e. affine linear maps from an abstract rational tropical curve with n marked points, i.e. a metric tree, to $\mathbb{R}^r$ such that the image is balanced around each vertex and is of degree Δ.

My research

I am trying to construct tropical varieties $\mathcal{M}_{0,n}(X,\Delta)$ parameterising rational tropical curves that map into X, where $X \subset \mathbb{R}^r$ is a smooth tropical variety, such that the space has the correct expected dimension. This space should be a natural subspace of $\mathcal{M}_{0,n}(\mathbb{R}^r,\Delta)$. In a joint work with Andreas Gathmann and Hannah Markwig we constructed such spaces in the case where X is itself a tropical curve. We consider the resolutions of a vertex of the underlying abstract tropical curve and glue the whole space $\mathcal{M}_{0,n}(X,\Delta)$ from these local patches by using tropical intersection theory.

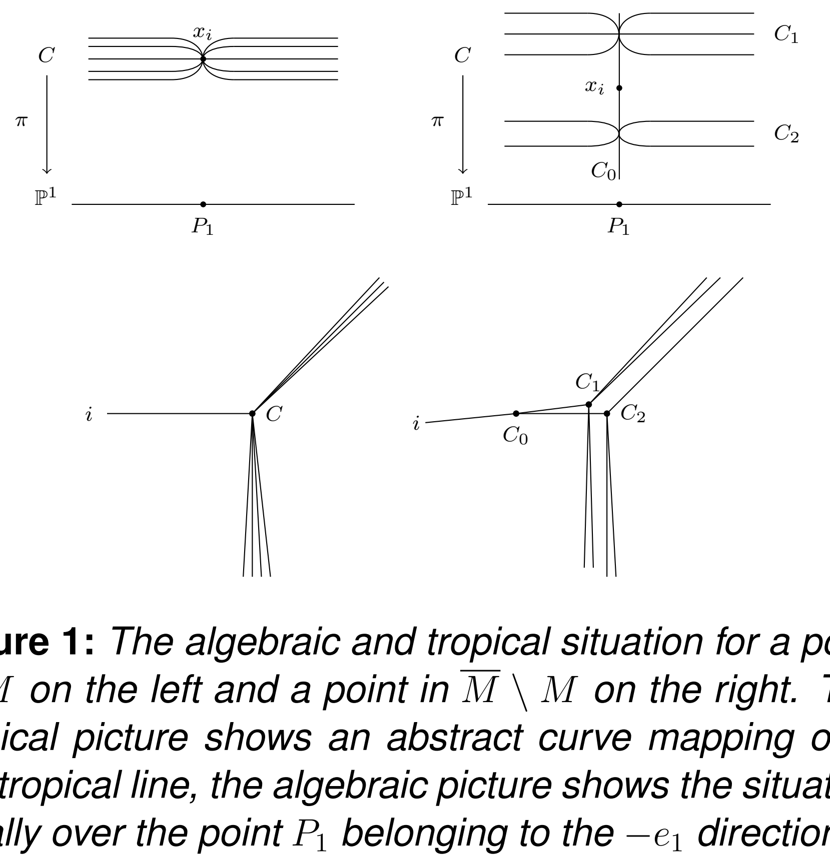

Figure 1: *The algebraic and tropical situation for a point in M on the left and a point in $\overline{M} \setminus M$ on the right. The tropical picture shows an abstract curve mapping onto the tropical line, the algebraic picture shows the situation locally over the point P_1 belonging to the $-e_1$ direction.*

This is possible since deformations of a tropical cover of a curve are "local" on the abstract curve, similar to the classical case of covers from $\mathbb{P}^1$ to $\mathbb{P}^1$. In fact we use much of the theory of the classical rational covers of $\mathbb{P}^1$ to obtain weights on our tropical space and to show balancing around codimension one cones. To obtain a space of the expected dimension, we impose a combinatorial condition on the valency of the abstract tropical curves. This condition also ensures, that there exist classical rational covers which have marked ramifications over points P_k with the profile given by the weights of the rays of the image of the abstract curve which have direction $-e_k$. Hence the marked points correspond to marked ends in the tropical curve. It turns out that we only have to consider curves for which the space M of such classical covers is one dimensional. Considering the closure $\overline{M}$ in $\overline{M}_{0,n}(\mathbb{P}^1,d)$, we obtain a finite number of reducible curves which correspond to the resolutions of the tropical vertex as in figure 1. Irreducible components of the classical cover correspond to vertices of the tropical resolution and the incidence relations between tropical vertices and algebraic components are the same. Using intersection theory on $\overline{M}$ yields tropical balancing with weights which are products of Hurwitz numbers.

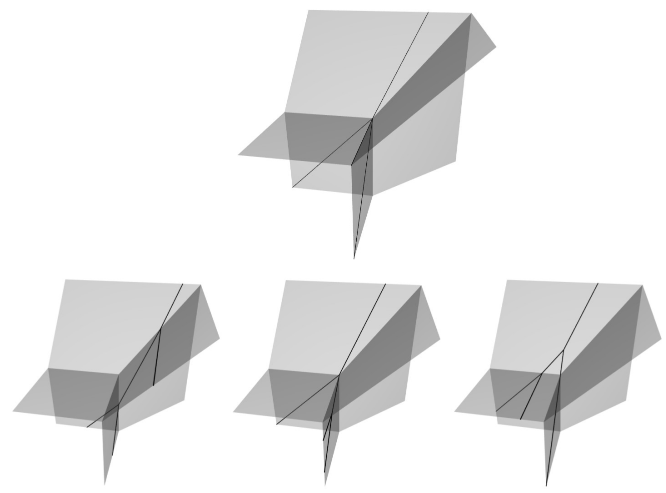

Figure 2: *Resolutions of a tropical curve in a plane*

I am currently trying to generalise this approach to smooth tropical hypersurfaces $X \subset \mathbb{R}^r$, but a lot of difficulties occur here, as neither the classical nor the tropical deformations are really local anymore. It is still possible to assign a space of classical curves in some $\overline{M}_{0,n}(\mathbb{P}^{r-1},d)$ to a tropical vertex in a way so that we obtain balanced tropical resolutions from intersection theory. However it is in unclear whether the classical space has the correct dimension and if the occuring weights are compatible to glue to a global space $\mathcal{M}_{0,n}(X,\Delta)$. This works out for the space of tropical lines in hypersurfaces of $\mathbb{R}^3$, where we obtain moduli spaces of the correct expected dimension $\deg X - 3$. In examples this moduli space for a cubic surface consists of 27 lines, but unfortunately we cannot prove this in general.