Frobenius-stable Lattices in p-adic Cohomology

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Problem statement

$$Z_{\overline{C}}(t) = \frac{\det\left(1 - \mathrm{Frob}\cdot t \mid \mathrm{H}^1_p(\overline{C})\right)}{(1-t)(1-qt)}$$

Given a smooth, projective, integral curve $\overline{C}/\mathbf{F}_q$ the Frobenius action on its first p-adic cohomology group completely determins its Zeta function. Can we find a natural, Frobenius-stable lattice inside $\mathrm{H}^1_p(\overline{C})$ together with

- an explicit description of the lattice
- a Frobenius action is "easy" to define and compute.

The generalised Bogaart's Lemma below gives such an explicit description of the log-crystalline cohomology and relates it to an integral version of Monsky-Washnitzer cohomology where a Frobenius action can be computed using a Newton iteration.

The map φ

$$\mathrm{H}^0(\Omega((\ell+1)D)) \xrightarrow{\varphi} \Upsilon = \bigoplus_j \bigoplus_{i=-(\ell+1)}^{2} (\mathbf{Z}_q/(i+1))\, t_j^i\, dt_j$$

The map sends a differential to the formal sum of its Laurent series expansions with respect to the t_j. Here, the t_j are rational functions on C such that $\{p, t_j\}$ is a coordinate system for $\mathcal{O}_{C,P_j}$ and P_j runs through all points in the support of D.

The isomorphism $\operatorname{Coker} d|_{d^{-1}(\operatorname{Ker}\varphi)} \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{dR}}(C,D)$

The idea is to compare the hypercohomology of $\Omega(\log D)^\bullet\colon \mathcal{O}_C \to \Omega(\log D) \to 0$ to the hypercohomlogy of $\widetilde{\Omega}^\bullet\colon \mathcal{O}_C(\ell D) \to \Omega((\ell+1)D)$. The latter complex consists of non-special sheaves, so it is easier to analyse. The long exact sequence to the short exact sequence $0 \to \Omega(\log D)^\bullet \to \widetilde{\Omega}^\bullet \to \mathcal{Q}^\bullet \to 0$ yields the following diagram with exact row and column and thus the desired isomorphism.

$$\begin{array}{ccccccccc}
 & & & & \mathrm{H}^0(\mathcal{O}_C(\ell D)) & & & & \\
 & & & & \downarrow d & & & & \\
 & & & & \mathrm{H}^0(\Omega((\ell+1)D)) & & & & \\
 & & & & \downarrow & \overset{\varphi}{\searrow} & & & \\
0 & \longrightarrow & \mathrm{H}^1_{\mathrm{dR}}(C,D) & \longrightarrow & \mathbf{H}^1(\widetilde{\Omega}^\bullet) & \longrightarrow & \Upsilon & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

A generalised Bogaart's Lemma [1]

Let $C/\mathbf{Z}_q$ be a smooth, proper lifting of $\overline{C}/\mathbf{F}_q$, D an effective normal crossings divisor $D \neq 0$, and $U = C\backslash D$. Let $d\colon \mathrm{H}^0(\mathcal{O}_C(\ell D)) \to H^0(\Omega((\ell+1)D))$ be the universal derivation. If ℓ is an integer such that $\mathcal{O}_{\overline{C}}(\ell\overline{D})$ is nonspecial, then there is the commutative diagram on the right hand side, functorial in (C,D), and with inclusions equivariant with respect to the Frobenius actions on $\mathrm{H}^1_{\mathrm{cr}}(\overline{C},\overline{D})$ and $\mathrm{H}^1_{\mathrm{MW}}(\overline{U})/(\mathrm{tor})$.

$$\begin{array}{ccccccc}
\operatorname{Ker}\varphi & \twoheadrightarrow & \operatorname{Coker} d|_{d^{-1}(\operatorname{Ker}\varphi)} & \xrightarrow{\sim} & \mathrm{H}^1_{\mathrm{dR}}(C,D) & \xrightarrow[{[3]}]{\sim} & \mathrm{H}^1_{\mathrm{cr}}(\overline{C},\overline{D}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^0(\Omega((\ell+1)D)) & \twoheadrightarrow & \operatorname{Coker} d/(\mathrm{tor}) & \hookrightarrow & \mathrm{H}^1_{\mathrm{dR}}(U)/(\mathrm{tor}) & \xrightarrow{\sim} & \mathrm{H}^1_{\mathrm{MW}}(\overline{U})/(\mathrm{tor})
\end{array}$$

What's next?

- Fleshing out the details in a chapter of my forthcoming thesis.
- Explaining how an explicit smooth, proper lifting of $\overline{C}$ can be approximated. (It's already implemented!)
- Using both to actually compute a basis of a Frobenius-stable lattice.

(The Frobenius action on) Monsky-Washnitzer cohomology [2]

If $\overline{A}/\mathbf{F}_q$ is the affine coordinate ring of $\overline{U}$ and $A/\mathbf{Z}_q$ is a smooth lifting, say $A = \mathbf{Z}_q[\underline{x}]/(\underline{f})$, then $\mathrm{H}^1_{\mathrm{MW}}(\overline{U}) = \left(\Omega_A \otimes A^\dagger\right)/dA^\dagger$, where $A^\dagger = \mathbf{Z}_q[\underline{x}]^\dagger/(\underline{f})$. The elements of $\mathbf{Z}_q[\underline{x}]^\dagger$ are formal power series in $\underline{x}$ over $\mathbf{Z}_q$ converging on a radius greater than 1. Roughly speaking, this allows Newton iteration style computations to lift the Frobenius action from A to $A^\dagger$ and $\Omega_A \otimes A^\dagger$.

References

[1] If D consist of a single $\mathbf{Z}_q$-point, then the top row of the diagram is Lemma 3.3.10 of T. van den Bogaart, *Links between cohomology and arithmetic*, PhD Thesis at Universiteit Leiden, 2008.

[2] P. Monsky and G. Washnitzer, *Formal Cohomology I*, Ann. of Math. (2) Vol. 88, 1968.

[3] K. Kato, *Logarithmic structures of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory, 1989.