$\text{Ш}(A/K)$ of non-square order

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The Tate-Shafarevich group

LET A be an abelian variety over a number field K. The Tate-Shafarevich group $\text{Ш}(A/K)$ of A over K can be defined in terms of Galois cohomology as

$$\text{Ш}(A/K) := \ker\left(H^1\left(G_K, A(\bar{K})\right) \to \prod_{\text{all } v} H^1\left(G_{K_v}, A(\bar{K}_v)\right)\right)$$

where in the product v runs over all (finite and infinite) places of K, and G_K and G_{K_v} are the absolute Galois groups of K and of its completion K_v at v.
It is known that $\text{Ш}(A/K)$ is an abelian torsion group. Geometrically, the non-trivial elements of $\text{Ш}(A/K)$ are those torsors (i.e. principal homogeneous spaces) of A/K for which the Hasse-principle fails, i.e. they have a point in any completion K_v, but not a rational point.

Conjecture 1. $\text{Ш}(A/K)$ *is a finite group.*

This important conjecture is related to the Birch and Swinnerton-Dyer conjecture and will underline all what follows. In case of an elliptic curve $E/\mathbb{Q}$, the conjecture is known to be true if the so called analytic rank of E is zero or one, which conjecturally holds for 100% of all elliptic curves over $\mathbb{Q}$.

Order of Ш

ASSUMING Conjecture 1, then for an elliptic curve E/K it is known that

$$\#\text{Ш}(E/K) = \square.$$

This is no longer true in higher dimensions. With the Cassels-Tate pairing, one can prove a necessary condition concerning the non-square part of the order of Ш.

Theorem 2. [Ste04] *Assuming Conjecture 1, if an odd prime p divides the non-square part of the order of $\text{Ш}(A/K)$, then p divides the degree of all polarizations of A/K.*

Corollary 3. [PS99] *Assuming Conjecture 1, then for a principally polarized abelian variety A/K we have*

$$\#\text{Ш}(A/K) = \square \text{ or } 2\square, \text{ (and both occur).}$$

A computation of Stein [Ste04] leads to the conjecture

Conjecture 4. *Assuming Conjecture 1, as one ranges over all abelian varieties A over all number fields K, then every non-square natural number can appear as the non-square part of the order of $\text{Ш}(A/K)$.*

So the natural question arising is

Question 5. *Assuming Conjecture 1, what are the possible non-square parts of the order of Ш for abelian varieties of fixed dimension over a fixed number field? Is this a finite list?*

An equation of Cassels and Tate

TO show the invariance under isogenies of the Birch and Swinerton-Dyer conjecture, Cassels (the elliptic curve case) and Tate (the general case) proved the following theorem.

Theorem 6. [Cas65] [Tat66] *Assuming Conjecture 1, let $\phi : A \to B$ be a K-isogeny between two abelian varieties A, B over a number field K. Then*

$$\frac{\#\text{Ш}(A/K)}{\#\text{Ш}(B/K)} = \frac{R_B P_B \# A(K)_{tor} \# A^\vee(K)_{tor}}{R_A P_A \# B(K)_{tor} \# B^\vee(K)_{tor}} \prod_{v \text{ bad}} \frac{c_{B,v}}{c_{A,v}}.$$

Here R_A and P_A are the regulator and period of A, which are complex transcendental numbers, but their quotient is rational. $A(K)_{tor}$ are the finitely many K-rational torsion points of A, $A^\vee$ is the dual abelian variety, and the $c_{A,v} \in \mathbb{N}$ are the local Tamagawa numbers of A at a finite place v of bad reduction.

My goals

ASSUMING Conjecture 1, I would like to give an answer to Question 5 for abelian surfaces over $\mathbb{Q}$.
For the moment I am trying to construct an example of $\#\text{Ш}(A/\mathbb{Q}) = 5\square$, with $\dim(A) = 2$. A strategy to get such an example is to start with a well understood principally polarized abelian surface A and an isogenous abelian surface B which fulfills Theorem 2 for $p = 5$. Then one tries to compute all quotients in the Cassels-Tate equation in Theorem 6 mod $\square$ to know $\text{Ш}(B/K)$ mod $\square$.

References

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