An algorithm to compute Segre classes

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Introduction

- Chern and Segre classes are important invariants in intersection theory and enumerative geometry.
- In [1]: Computation of the *degrees* of the Chern classes for smooth subvarieties of complex projective space. (See definition of degree below.)
- We use a similar method to compute the degrees of the Segre classes of closed subschemes of complex projective space.

Setting

Let Y be an n-dimensional closed subscheme of $\mathbb{P}^k(\mathbb{C})$. The Segre classes $s_0(Y,\mathbb{P}^k)$, …, $s_n(Y,\mathbb{P}^k)$ of Y in $\mathbb{P}^k$ are equivalence classes of cycles in Y modulo rational equivalence, hence they can be written as the weighted sum of subvarieties of Y. The degree is defined as the weighted sum of the degrees of these subvarieties.

Definition of Segre classes

Let $\widetilde{\mathbb{P}^k}$ be the blow-up $\mathrm{Bl}_Y\mathbb{P}^k$ of $\mathbb{P}^k$ along Y, $\widetilde{Y}$ the exceptional divisor and $\eta : \widetilde{Y} \to Y$ the projection. Then the total Segre class of Y in $\mathbb{P}^k$ is defined as

$$s(Y,\mathbb{P}^k) = \sum_{p\geq 1}(-1)^{p-1}\eta_*(\widetilde{Y}^p),$$

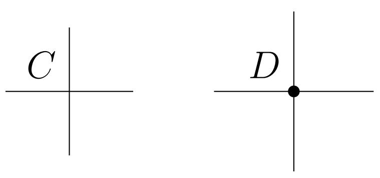
where $\widetilde{Y}^p$ denotes the p-fold self-intersection. The i-th Segre class $s_i(Y,\mathbb{P}^k)$ is the codimension i part of the total Segre class $s(Y,\mathbb{P}^k)$.

Toy example: smooth curves

Let Y be a smooth curve. Then the degree of the first Chern class $c_1(Y)$ is the topological Euler characteristic $\chi = 2-2g$ of Y and the degree of $c_0(Y)$ is the degree of Y. The Segre classes can be computed from the Chern classes, and vice versa. Hence for smooth curves, Segre classes carry the same information as degree and genus.

Example: detecting embedded points

We work with singular curves in the projective plane $\mathrm{Proj}\,\mathbb{C}[x,y,z]$. Consider the scheme C given by $I_C = (xy)$ and the scheme D given by $I_D = (x^2y, xy^2)$. They have the same underlying topological space, but C is reduced and D has an embedded point.

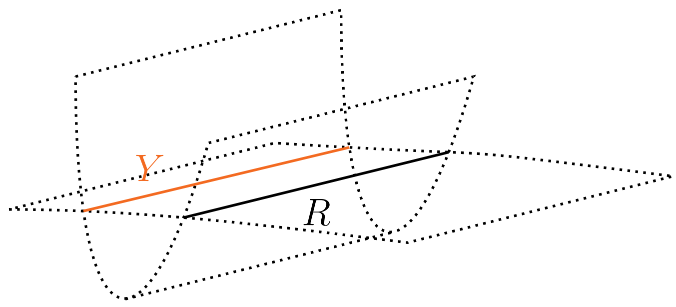


The degrees of the Segre classes detect the embedded point:

	ideal	$\deg s_1(-,\mathbb{P}^2)$	$\deg s_0(-,\mathbb{P}^2)$
C	(xy)	-4	2
D	(x^2y, xy^2)	-3	2

Idea of the algorithm

- Choose hypersurfaces containing Y, they intersect in Y and a residual scheme R.



- Prove Bézout-like theorem relating degrees of the Segre classes to the degree of the residual.
- Compute residual with symbolical methods (saturation of ideals) or numerical methods.

Bézout-like theorem

(Eklund, J., Peterson) Let $Y \subset \mathbb{P}^k$ be a subscheme of dimension n defined by a non-zero homogeneous ideal $I \subseteq \mathbb{C}[x_0,\ldots,x_k]$. Let $g_0,\ldots,g_r$ be a set of non-zero homogeneous generators of I and put $m = \max_i\{\deg g_i\}$. For $k-n \leq d \leq k$ and general elements $f_1,\ldots,f_d \in I$ of degree m, the following holds: If J is the ideal generated by $\{f_1,\ldots,f_d\}$ and $R \subseteq \mathbb{P}^k$ is the subscheme defined by $J : I^\infty$, then

$$m^d = \deg R + \sum_{i=0}^{p}\binom{d}{p-i}m^{p-i}\deg s_i(Y,\mathbb{P}^k),$$

where $p = d-(k-n)$.

Implementation

- Test implementation on my homepage `www.math.su.se/~jost`
- In preparation: implementation in Macaulay2 using symbolic methods (Gröbner bases)
- Also in preparation: implementation in Bertini and Macaulay2 using numerical methods (homotopy continuation)

References

[1] S. Di Rocco, D. Eklund, C. Peterson, and A. J. Sommese. Chern numbers of smooth varieties via homotopy continuation and intersection theory. *J. of Symb. Comp.*, 46(1):23 – 33, 2011.