Russell's hypersurface

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Russell's hypersurface and the linearization conjecture

RUSSELL'S hypersurface X is defined by

$$X = V(x + x^2y + z^3 + t^2) \subset \mathbb{C}^4.$$

It satisfies $X \overset{\text{diff}}{\cong} \mathbb{R}^6$, but does also $X \overset{\text{iso}}{\cong} \mathbb{C}^3$ hold? If so, this would imply a counterexample to the linearization conjecture, which states that every $\mathbb{C}^*$-action on $\mathbb{C}^n$ is linearizable (i.e. that it becomes linear after a polynomial change of variables on $\mathbb{C}^n$). Namely, the action

$$\begin{array}{ccc} \mathbb{C}^* \times X & \longrightarrow & X \\ (\lambda, (x, y, z, t)) & \mapsto & (\lambda^6 x, \lambda^{-6} y, \lambda^2 z, \lambda^3 t) \end{array}$$

restricts to an action of the group of sixth roots of unity $C_6 \times X \longrightarrow X$ with invariant set

$$X^{C_6} = V(\mathbb{C}^2; x(1 + xy)) \times 0 \times 0 \cong \mathbb{C} \,\dot{\cup}\, \mathbb{C}^*$$

This is not connected, and therefore the induced action on X is not linear.
The linearization conjecture has been shown for $n \leq 3$, using Makar-Limanov's result that

$$\mathrm{ML}(X) := \bigcap_{\text{all } \mathbb{C}^+\text{-actions}} \mathcal{O}(X)^{\mathbb{C}^+} \neq \mathbb{C} = \mathrm{ML}(\mathbb{C}^3)$$

– in particular $X \ncong \mathbb{C}^3$.

A way of calculating $\mathrm{ML}(X)$

THE reson that $\mathrm{ML}(\mathbb{C}^3) = \mathbb{C}$ is that there are no nonconstant polynomials in three variables that are invariant with respect to all $\mathbb{C}^+$-actions (e.g. translations) on $\mathbb{C}^3$. Calculating $\mathrm{ML}(X)$, and being able to distinguish it from $\mathbb{C}$, is harder but it can be done by showing that the coodinate function $x \in \mathcal{O}(X)$ is an invariant with respect to all $\mathbb{C}^+$-actions on X. This implies that the $\mathbb{C}$-algebra $\mathrm{ML}(X)$ has transcendence degree at least one. An argument goes like follows: X can be realized as an open part of a blowup M of $\mathbb{C}^3$ with center at the Niel parabola $N := V(\mathbb{C}^3; x^2, x + z^3 + t^2) \subset \{0\} \times \mathbb{C}^2$

$$\begin{array}{ccccc} \pi^{-1}(U) & \longrightarrow & M & \longleftarrow & N \times \mathbb{P}_1 \\ \downarrow \cong & & \downarrow \pi & & \downarrow \pi \\ U := \mathbb{C}^3 \setminus N & \longrightarrow & \mathbb{C}^3 & \longleftarrow & N \end{array}$$

and it turns out that the result $x \in \mathrm{ML}(X)$ follows from the fact that every $\mathbb{C}^+$-action on X descends to $\mathbb{C}^3$. This in turn follows from the fact that any nontrivial $\mathbb{C}^+$-action on X induces a nontrivial $\mathbb{C}^+$-action on $W := \mathrm{Sp}(B)$, where B is the graded algebra associated to the filtration of $\mathcal{O}(X)$ given by the pole order along the divisor $M \setminus X$ at infinity, and that the degree k of homogeneity of its corresponding locally nilpotent derivation is negative.

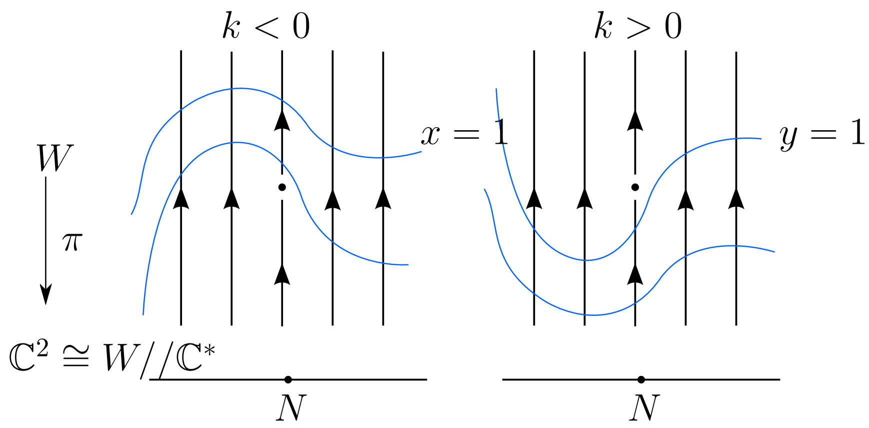

Figure 1: *Quotient projections of the canonical $\mathbb{C}^*$-action on W with $k < 0$ resp. $k > 0$.*

My research

AMONG my research interests are mainly topics in affine algebraic geometry: The linearization conjecture, Algebraic group actions on varieties, Invariant theory, The Makar-Limanov invariant, Locally nilpotent derivations. Also: Is Russell's hypersurface analytically isomorphic to $\mathbb{C}^3$?