Rational points of varieties over $\mathbb{C}((t))$

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Motivation

WHICH class of algebraic varieties over a given field automatically has a rational point?
This is a question of current research. For example there is the following result in [1]:

Theorem. *Every projective rationally connected variety X over $\mathbb{C}(t)$ has a rational point.*

The same is true for $K := \mathbb{C}((t))$. But for example it is not known what happens if we weaken the assumption on X slightly.

HOW can one detect rational points of a given K-variety X?
The idea is to look at a model $\mathcal{X}$ of X, i.e. an S-variety, $S :=$ Spec$(\mathcal{O}_K)$, whose generic fiber is isomorphic to X. If $\mathcal{X}$ is proper over S, we get the following one-to-one correspondence:

$$K\text{-points of } X \xleftrightarrow{1:1} S\text{-points of } \mathcal{X}$$

To find an S-points of $\mathcal{X}$, look at the special fiber $\mathcal{X}_\mathbb{C} := \mathcal{X} \times_S$ Spec$(\mathbb{C})$ of $\mathcal{X}$. We have a reduction map $\phi : \mathcal{X}(S) \to \mathcal{X}_\mathbb{C}(\mathbb{C})$. As $\mathcal{O}_K$ is Henselian, we have Sm$(\mathcal{X}_\mathbb{C})(\mathbb{C}) \subset \phi(\mathcal{X}(S))$. If $\mathcal{X}$ is regular, we even get an equality, so in this case $\mathcal{X}$ has an S-point if and only if $\mathcal{X}_\mathbb{C}$ has a reduced component. In general it is not known if a given singular point $x \in \mathcal{X}_\mathbb{C}$ lies in the image of ϕ.
To find a nice model of X it is sometimes easier to look for a nice model $\mathcal{Y}$ of $X_L := X \times_{\text{Spec}(K)}$ Spec(L), with L over K a finite field extension, for example one can demand the special fiber of $\mathcal{Y}$ to be a reduced simple normal crossing divisor (semistable reduction).
In order to be able to construct a model $\mathcal{X}$ of X from a model $\mathcal{Y}$ of X_L, assume that the diagonal action of $G :=$ Gal(L/K) on $X_L \subset \mathcal{Y}$ extends to a G-action on $\mathcal{Y}$, which is compatible with the G-action on $T :=$ Spec$(\mathcal{O}_L)$ given by restriction of the Galois action of G on L to $\mathcal{O}_L \subset L$. Then $\mathcal{X} := \mathcal{Y}/G$ is an $S = T/G$-scheme with generic fiber X, i.e. a model of X.
The aim of my work is to understand a model of a K-variety X constructed as described above.

My Research

FIX a regular T-scheme $\mathcal{Y}$ with a G-action which is compatible with the G-action on T. Let Sm$(\mathcal{Y}) \subset \mathcal{Y}$ be the smooth locus of the structure map of $\mathcal{Y}$.

Proposition. *$\mathcal{X} := \mathcal{Y}/G$ has an S-point if and only if there exists a closed fixed point $x \in$ Sm$(\mathcal{Y})$.*

The most interesting part of the proof is the "if"-part, done by constructing an S-point of $\mathcal{X}/G$ through the image of the fixed point $x \in$ Sm(X) under the quotient map. This is done by constructing a G-invariant T-point of $\mathcal{X}$ through x using the fact that the G-action is diagonalizable on the complete local ring and a kind of weighted, G-equivariant blowup.
Note that the constructed S-point might be in the singular locus of $\mathcal{X}/G$. To see this look at the following example:

Example. Look at the $G = \mathbb{Z}/2\mathbb{Z}$-action on the smooth T-scheme $\mathbb{A}^1_T =$ Spec$(\mathbb{C}[\![u]\!][x])$ given by
$g : \mathbb{C}[\![u]\!][x] \to \mathbb{C}[\![u]\!][x]; P(u, x) \to P(-u, -x)$
This action is compatible with the G-action on T, given by $g_T : \mathbb{C}[\![u]\!] \to \mathbb{C}[\![u]\!]; P(u) \to P(-u)$. The closed point $Q = (0, 0)$ is fixed.
$\mathbb{A}^1_T/G \cong \mathbb{C}[[t]][b, c]/(tb - c^2)$ is singular in the image $Q' = (0, 0, 0)$ of Q under the quotient map.

FOR a given K-variety X one can define the Serre invariant as follows: One takes a weak Néron model $\mathcal{X}$ of X, i.e. $\mathcal{X}$ is smooth over S and $\mathcal{X}(S) = X(K)$. Such a model always exists, but is not unique. The Serre invariant $S(X)$ is the special fiber of $\mathcal{X}$ in $K_0(\text{Var}_\mathbb{C})/(\mathbb{L} - 1)$, which is independent of the choice of a weak Néron model. Here $K_0(\text{Var}_\mathbb{C})$ is the Grothendieck Ring of varieties, generated by isomorphism classes $[Z]$ of separated $\mathbb{C}$-schemes of finite type and for every closed immersion $Z' \to Z$ relations $[Z] = [Z \setminus Z'] + [Z']$, with multiplication given by fiber product over $\mathbb{C}$, and $\mathbb{L} = [\mathbb{A}^1_\mathbb{C}]$. For the construction see [2].
Now take a finite field extension L of K and a model $\mathcal{Y}$ of X_L with a G-action as above. Let $\mathcal{Y}^G \subset \mathcal{Y}$ be the fixed point locus of this action. By the Proposition we have that $S(X) = 0$ if and only if $[\mathcal{Y}^G] = 0 \in K_0(\text{Var}_\mathbb{C})/(\mathbb{L} - 1)$.

Question. *Do we have $S(X) = [\mathcal{Y}^G] \in K_0(\text{Var}_\mathbb{C})/(\mathbb{L}-1)$?*

References

[1] Tom Graber, Joe Harris, and Jason Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67 (electronic). MR 1937199 (2003m:14081)

[2] Johannes Nicaise, *A trace formula for varieties over a discretely valued field*, J. Reine Angew. Math. **650** (2011), 193–238. MR 2770561