Stability conditions on Kummer surfaces

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1 Moduli spaces of SCFTs

Mirror symmetry and Bridgeland's stability conditions are mathematical theories motivated by superconformal field theories (SCFT) associated to Calabi-Yau varieties. It is a non-trivial task to give a rigorous geometric interpretation of a SCFT. This is understood in the case of complex tori [1] and progress has been made for certain K3 surfaces [2] and Borcea-Voisin 3-folds (Khalid/Wendland) using realizations by non-linear σ models of orbifold conformal field theories.

A $N=(2,2)$ superconformal field theory is roughly a conformal field theory with the left- and right-handed Virasoro algebra extended to commuting N=2 super-Virasoro algebra and a $\mathbb{Z}$ grading $\mathscr{H}=\mathscr{H}_B\oplus\mathscr{H}_F$. In general, we have another compatible $\mathbb{Z}_2\times\mathbb{Z}_2$ grading $\mathscr{H}=\mathscr{H}_{NS,NS}\oplus\mathscr{H}_{R,R}\oplus\mathscr{H}_{NS,R}\oplus\mathscr{H}_{R,NS}$.
A toroidal CFT with central charges $c=\bar{c}=3$ is uniquely determined by its charge lattice $\Gamma\subset\mathbb{R}^{2,2}$ with Euclidean scalar product. Each $p,\bar{p}\in\Gamma$ labels a vertex operator of same charge with respect to the real and imaginary part of bosonic currents arising from the definition. The charge lattice can be defined by two moduli $\tau,\rho\in\mathbb{H}$. The parameter $\tau\in\mathbb{H}$ is interpreted as the period of the elliptic curve E_τ, fixing its complex structure and the parameter $\tau\in\mathbb{H}$ gives the Kahler structure as $\Im(\tau)>0$ is the volume of the elliptic curve E_τ. The moduli space of SCFTs associated to elliptic curves is

$$\mathscr{M}=(PSL_2(\mathbb{Z})\backslash\mathbb{H}\times PSL_2(\mathbb{Z})\backslash\mathbb{H})/\mathbb{Z}_2^2.$$

Two-dimensional Calabi-Yau manifolds are complex tori and K3 surfaces. The moduli space of SCFTs with central charge c=6 is given by the following

Theorem 1.1. *[3] Every connected component of the moduli space of SCFTs associated to Calabi-Yau 2-folds is either of the form $\mathscr{M}_{tori}=\mathscr{M}_0$ or $\mathscr{M}_{K3}=\mathscr{M}_{16}$ where:*

$$\mathscr{M}^\delta\cong O^+(4,4+\delta;\mathbb{Z})\backslash O^+(4,4+\delta;\mathbb{R})/SO(4)\times O(4+\delta).$$

Points $x\in\tilde{\mathscr{M}}^\delta$ in the Grassmannian $\tilde{\mathscr{M}}^\delta=O^+(4,4+\delta;\mathbb{R})/SO(4)\times O(4+\delta)$ correspond to positive definite oriented four-planes in $\mathbb{R}^{4,4+\delta}$. Let $x\subset H^{even}(X,\mathbb{R})$ be a positive oriented four-plane specifying a SCFT on X. A geometric interpretation of this SCFT is a choice of null vectors $\upsilon^0,\upsilon\in H^{even}(X,\mathbb{Z})$ along with a decomposition of x into two perpendicular oriented two-planes $x=\Omega\perp\mho$ such that $\langle v^0,\upsilon^0\rangle=\langle\upsilon,\upsilon\rangle=0$, $\langle\upsilon^0,\upsilon\rangle=1$, and $\Omega\perp\upsilon^0,\upsilon$.

Lemma 1.2. *[3] Let $x\subset H^{even}(X,\mathbb{R})$ be a positive oriented four-plane with geometric interpretation $\upsilon^0,\upsilon\in H^{even}(X,\mathbb{Z})$, which are interpreted as generators of $H^0(X,\mathbb{Z})$ and $H^4(X,\mathbb{Z})$, respectively, and a decomposition $x=\Omega\perp\mho$. Then one finds uniquely $\omega\in H^{even}(X,\mathbb{R})$ (up to scaling) and $B\in H^2(X,\mathbb{R})$ with*

$$\mho=\mathbb{R}\left\langle\omega-\langle\omega,B\rangle,\xi_4=\upsilon^0+B+\left(V-\frac{1}{2}\langle B,B\rangle\right)\upsilon\right\rangle$$

with $V\in\mathbb{R}_+$ and $\omega^2\in\mathbb{R}_+$. It is important to note that the mentioned moduli space of SCFTs associated to K3 surfaces also contains ill-defined conformal field theories. Namely, a positive oriented four-plane $x\in\tilde{\mathscr{M}}^{16}$ corresponds to such an theory if and only if there is a $\delta\in x^\perp\subset\mathbb{R}^{4,20}$ with $\langle\delta,\delta\rangle=-2$. The physics behind this tells us that the field theory gets extra massless particles at certain points in the moduli space and breaks down.

2 Orbifold conformal field theories

Given a finite group G that acts nicely on the Hilbert space of a SCFT one can construct an orbifold conformal field theory by projecting onto G invariant representations of the superconformal algebra and adding so-called twisted representations. We are interested in the case of a complex torus T with $\mathbb{Z}_2$ action and its associated Kummer surface X. The twisted representations correspond to the exceptional divisors $E_i, i=1,\ldots,26$ of the blow up. Let $\pi:T\to X$ be the induced rational map of degree $|G|$ defined outside the fixed points of the G action. The induced map on the cohomology gives an embedding into a lattice that is isometric to the even cohomology $H^{even}(X,\mathbb{Z})$: $\pi_*:H^{even}(T,\mathbb{Z})(2)\hookrightarrow L(X)$ [4, 2].

Theorem 2.1. *[2] For a geometric interpretation of a N=(2,2) SCFT on a complex torus T with ω,V_T,B_T as in lemma (1.2) the corresponding orbifold conformal field theory has a geometric interpretation $\hat{\upsilon}$, $\hat{\upsilon}^0\in H^{even}(X,\mathbb{Z})$ with $\pi_*\omega,V=\frac{V_T}{2},B$ where*

$$B=\frac{1}{2}\pi_*B_T+\frac{1}{2}B_{\mathbb{Z}},B_{\mathbb{Z}}=\frac{1}{2}\sum_{i\in\mathbb{F}_2^4}\hat{E}_i.$$

$x^\perp\cap H^{even}(X,\mathbb{Z})$ does not contain (-2) classes.

Here $\pi_*\omega$ is an orbifold Kahler class on the boundary of the Kahler cone.

3 Bridgeland stability conditions

Definition 3.1. [5] A *stability condition on a triangulated category $\mathscr{D}$* consists of a group homomorphism $Z:K(\mathscr{D})\to\mathbb{C}$ called the *central charge* and of full additive subcategories $\mathscr{P}(\phi)\subset\mathscr{D}$ for each $\phi\in\mathbb{R}$, satisfying the following axioms:

1. if $0\neq E\in\mathscr{P}(\phi)$, then $Z(E)=m(E)exp(i\pi\phi)$ for some $m(E)\in\mathbb{R}_{>0}$;

2. $\forall\phi\in\mathbb{R},\mathscr{P}(\phi+1)=\mathscr{P}(\phi)\,[1]$;

3. if $\phi_1>\phi_2$ and $A_j\in\mathscr{P}(\phi_j)$, then $Hom_{\mathscr{D}}(A_1,A_2)=0$;

4. for $0\neq E\in\mathscr{D}$, there is a finite sequence of real numbers $\phi_1>\cdots>\phi_n$ and a collection of triangles

$$E_{i-1}\longrightarrow E_i\longrightarrow A_i$$

with $E_0=0$, $E_n=E$ and $A_j\in\mathscr{P}(\phi_j)$ for all j.

Example: Let C be a smooth projective curve over $\mathbb{C}$. The central charge $Z(E)=-deg(E)+irk(E)$ is a stability function on the Abelian category of coherent sheaves on C $Coh(C)$ with the HN property. There is only one stability condition up to the action of $\widetilde{GL^+(2,\mathbb{R})}$ which is free and transitive on Stab(E) [5, 6]. Thus

$$Stab(C)\cong\widetilde{GL^+(2,\mathbb{R})}\cong\mathbb{C}\times\mathbb{H}.$$

For an elliptic curve E we find

$$Aut(E)\backslash Stab(E)/\mathbb{C}\cong\mathbb{H}/PSL(2,\mathbb{Z})$$

where $Aut(E)$ is the group of auto equivalences of $D^b(E)$. If we compare this to the moduli space of SCFTs associated to elliptic curves we can interpret the Kahler moduli space of the elliptic curve as a subspace of this quotient.

4 Stability conditions on K3 surfaces

We are interested in the bounded derived category of coherent sheaves $D^b(X)$ on a projective K3 surface or an Abelian surface X over the complex numbers. In this case we say a stability condition is numerical if the central charge $Z:K(X)\to\mathbb{C}$ factors through the quotient group $\mathscr{N}(X)=K(X)/K(X)^\perp=\mathbb{Z}\oplus NS(X)\oplus\mathbb{Z}$.

Theorem 4.1. *[5] For each connected compenent $Stab^*(X)\subset Stab(X)$, there is a linear subspace $V\subset\mathscr{N}(X)\otimes\mathbb{C}$ such that*

$$\pi:Stab^*(X)\longrightarrow\mathscr{N}(X)\otimes\mathbb{C}$$

is a local homeomorphism onto an open subset of the subspace V. In particular, $Stab^*(X)$ is a finite-dimensional complex manifold.

We have the following description of the stability manifold for algebraic K3 surfaces:

Theorem 4.2. *[7] There is a distinguished connected component $Stab^\dagger(X)\subset Stab(X)$ which is mapped by π onto the open subset $\mathscr{P}_0^+(X)$.*

The induced map $\pi:Stab(X)\to\mathscr{P}_0^+(X)$ is a covering map, and the subgroup of $Aut^0(\mathscr{D}(X))$ which preserves the connected component $Stab^\dagger(X)$ acts freely on $Stab^\dagger(X)$ and is the group of deck transformations of π.

Here $\mathscr{P}_0^+(X)$ is one component of the vectors in $\mathscr{N}\otimes\mathbb{C}$ whose real and imaginary part span positive 2-planes without ill-defined CFTs.

We noticed that the orbifold conformal field theories coming from a complex torus are all well-defined. For an Abelian surface and the Kummer surface $X=$ Km A we have an embedding $\pi_*:\mathscr{P}^+(A)\hookrightarrow\mathscr{P}_0^+(X)$. This can be liftet to the stability manifold:

Theorem 4.3. *Let $Stab^\dagger(A)$ the unique maximal connected component $Stab^\dagger(A)$ of the space of stability conditions of an Abelian surface A and $Stab^\dagger(X)$ the distinguished connected component of Stab(x) of the Kummer surface X=KM A. Then there is an embedding $Stab^\dagger(A)\hookrightarrow Stab^\dagger(X)$.*
Furthermore we have an injective homomorphism from the group of Deckbewegungen of $Stab^\dagger(A)$ (generated by the double shift) to the group of Deckbewegungen of $Stab^\dagger(X)$.

Question: Is $Stab^\dagger(X)$ for a projective Kummer surface simply connected? The positive answer would provide information on the group of auto equivalences of $D^b(X)$.

References

[1] K. NARAIN: New Heterotic String Theories in uncompactified Dimensions <10, *Phys. Lett.* 169B(1986), 41-46.

[2] W. NAHM & K. WENDLAND: A Hiker's guide to K3: Aspects of N=(4,4) Superconformal Field Theory with central charge c=6, *Comm. Math. Phys.* 216(2001),85-138.

[3] P. ASPINWALL & D. MORRISON: String Theory on K3 Surfaces, *in: Mirror Symmetry II (B. Greene and S.-T. Yau, eds.), Cambridge* 1997, 703-716.

[4] V. NIKULIN: On Kummer Surfaces, *Math. USSR Isv.* 9(1975), 261-275.

[5] T. BRIDGELAND: Stability Conditions on Triangulated Categories, *Ann. of Math.*, 166(2), 2007, 317-345.

[6] E. MACRI: Stability Conditions on Curves, *Math. Res. Lett.* , 14 (2007) 657-672.

[7] T. BRIDGELAND: Stability Conditions on K3 Surfaces, *Duke Math. J.*, 141(2), 2008, 241-291.