The Segre conjecture and the Mori Cone of a blown up surface

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Field of interest and the problem

BIRATIONAL geometry, Mori theory and algebraic surfaces. We are interested in the shape of the Mori cone for algebraic surfaces and we study the influence of the **Segre Conjecture** on the **Mori Cone** of a surface Y blown up at $r \gg 1$ general points. In particular we show that this conjecture implies the existence an $\mathbb{R}$-divisor D such that the $D^{\geqslant 0}$ part of $\overline{\mathrm{NE}}(X)$ does coincide with the $D^{\geqslant 0}$ part of the **positive cone** $\overline{\mathrm{Pos}}(X)$. It is a generalization of some results in the preprint *On the mori cone of blow-ups of the plane* by Tommaso de Fernex.

Setting

Y be an algebraic projective surface.

$x_1, \dots, x_r$ be r points on Y in general position.

$X = \mathrm{Bl}_r\, Y$ be the blow up of Y at $x_1, \dots, x_r$; $K = K_X$.

If L is a line bundle on X, $\mathcal{L}$ is the associated linear system.

Main characters

$\overline{\mathrm{NE}}(X) = \overline{\{\text{effective 1-cycles on } S\}}$.

$\overline{\mathrm{Pos}}(X) = \{\alpha \in N^1(X) \mid \alpha^2 \geqslant 0, \alpha \cdot h \geqslant 0 \text{ for } h \text{ ample}\}$.

Linear systems

Virtual dimension: $v(\mathcal{L}) = \chi(L) - 1$.

Expected dimension: $e(\mathcal{L}) = \max\{v(\mathcal{L}), -1\}$.

$\mathcal{L}$ is special if $\dim(\mathcal{L}) > e(\mathcal{L})$.

$\mathcal{L}$ is non special if $\dim(\mathcal{L}) = e(\mathcal{L})$.

Segre conjecture

SHGH Conjecture. *Let X the blow up at r general points of Y and let $\mathcal{L}$ be a non empty, non exceptional and reduced linear system; then $\mathcal{L}$ is not special.*

If $Y = \mathbb{P}^2$ and $r \geqslant 2$, this implies the existence of the decomposition

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{Pos}}(X) + \sum R(C), \quad C\ (-1)\text{-curve}.$$

Hint: locate curves such that $C^2 < 0$

The conjecture is false for Y abelian, Enriques or bi-ielliptic: we focus on Y a $K3$ surface or a surface with geometric genus $p_g(Y) = 0$.

List conjecture

$C \subset X$ is a $(-n, p)$-curve if $C^2 = -n, p_a(C) = p$.

List conjecture. There exist $\nu > 0$, $\pi \geqslant 0$ s.t. if $C^2 < 0$, then C is a $(-n, p)$-curve for some $0 < n \leqslant \nu$, $0 \leqslant p \leqslant \pi$.

Segre conjecture $\Rightarrow$ List conjecture with $\nu = \chi(\mathcal{O}_Y)$, $\pi = \chi(\mathcal{O}_Y) - 1$.

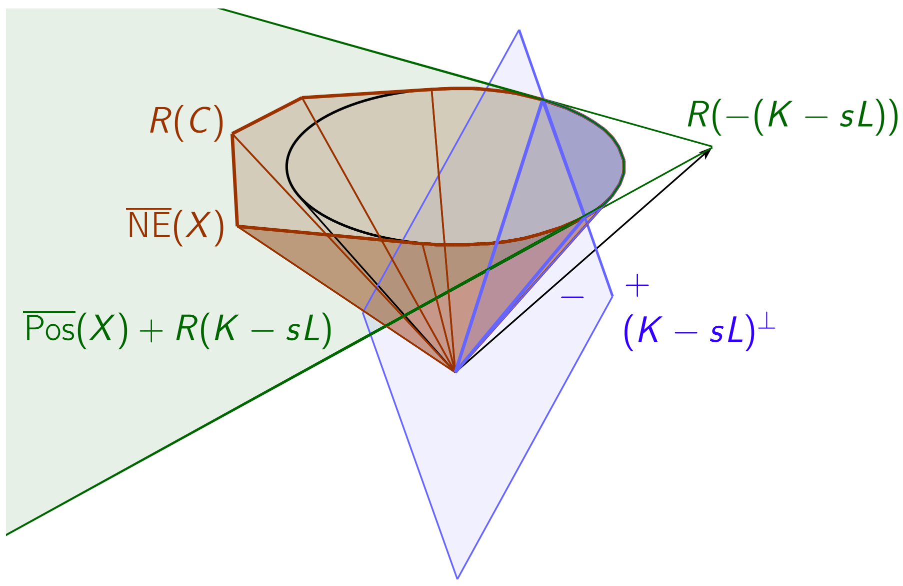
The result

Theorem. *Let $X = \mathrm{Bl}_r\, Y$; let A be ample on Y and L its pullback. Let us suppose the list conjecture holds true on X with bounds depending on Y and r sufficiently large (explicit inequalities depending on ν and π). Then there exists an $s \in \mathbb{R}$ (explicit) such that:*

$$\overline{\mathrm{NE}}(X)_{(K-sL)^{\geqslant 0}} = \overline{\mathrm{Pos}}(X)_{(K-sL)^{\geqslant 0}}.$$

In particular, if Y is either a $K3$ surface or such that $p_g(Y) = 0$ and the Segre conjecture holds true, then the $\overline{\mathrm{NE}}(X)$ cone has a circular part.

The picture



The strategy

1. For each $n = 1, \dots, \nu$ find s_n such that if C is $(-n, p)$-curve, then $R(C) \subset \overline{\mathrm{Pos}}(X) + R(K - s_n L)$.
2. All the rays with negative self intersection lie in $\overline{\mathrm{Pos}}(X) + R(K - sL)$, where $s = s_\nu$.
3. $R(C)^{\perp} \cap \overline{\mathrm{Pos}}(X) \subseteq (K - sL)^{\leqslant 0}$.
4. $\partial\, \mathrm{Nef}(X)_{(K-sL)^{>0}} = \partial \overline{\mathrm{Pos}}(X)_{(K-sL)^{>0}}$.
5. $\mathrm{Nef}(X)_{(K-sL)^{\geqslant 0}} = \overline{\mathrm{Pos}}(X)_{(K-sL)^{\geqslant 0}}$.
6. $\overline{\mathrm{NE}}(X)_{(K-sL)^{\geqslant 0}} = \overline{\mathrm{Pos}}(X)_{(K-sL)^{\geqslant 0}}$.