The tropical Torelli map

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This poster presents a combinatorial and computational study of the tropical moduli spaces M_g^{tr} and A_g^{tr} and the tropical Torelli map.

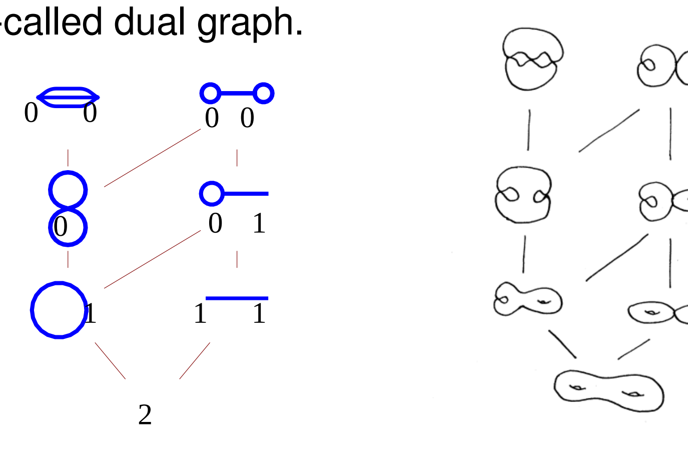
There is, of course, a vast literature on the subjects of algebraic curves and moduli spaces in algebraic geometry. For example, two well-studied objects are the moduli space $\mathcal{M}_g$ of smooth projective complex curves of genus g and the moduli space $\mathcal{A}_g$ of g-dimensional principally polarized complex abelian varieties. The Torelli map

$$t_g : \mathcal{M}_g \to \mathcal{A}_g$$

then sends a genus g algebraic curve to its Jacobian. The image of t_g is called the Torelli locus or the Schottky locus. The problem of how to characterize the Schottky locus inside $\mathcal{A}_g$ is already very deep.

The perspective we take here is the perspective of **tropical geometry**. From this viewpoint, one replaces algebraic varieties with piecewise-linear or polyhedral objects. These latter objects are amenable to combinatorial techniques, but they still carry information about the former ones. Roughly speaking, the information they carry has to do with what is happening "at the boundary" or "at the missing points" of the algebraic object.

For example, the tropical analogue of $\mathcal{M}_g$, denoted M_g^{tr}, parametrizes certain weighted metric graphs, and it has a poset of cells corresponding to the boundary strata of the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ of $\mathcal{M}_g$. Under this correspondence, a stable curve C in $\overline{\mathcal{M}_g}$ is sent to its so-called dual graph.



(a) (b)

Figure 1: *Posets of cells of tropical (a) and classical (b) moduli spaces of genus 2 curves. Vertices in (a) record irreducible components in (b), weights record genus, edges record nodes.*

We remark that the correspondence above yields dual graphs that are just graphs, not metric graphs. One can refine the correspondence using Berkovich analytification, whereby an algebraic curve over a complete nonarchimedean valued field is associated to its Berkovich skeleton, which is intrinsically a metric graph. In this way, one obtains a map between classical and tropical moduli spaces.

Our starting point is the recent paper by Brannetti, Melo, and Viviani [1]. There, the authors define the tropical versions M_g^{tr} and A_g^{tr} of $\mathcal{M}_g$ and $\mathcal{A}_g$, and a tropical Torelli map between them. In the paper [2] presented here, we do a combinatorial and computational study.

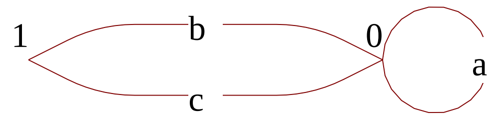


Figure 2: *This is an example of a* **tropical curve** *of genus 3. Here, a, b, c are fixed positive real numbers. Its* **genus** *is the number of independent cycles in the graph, plus the sum of all the vertex weights.*

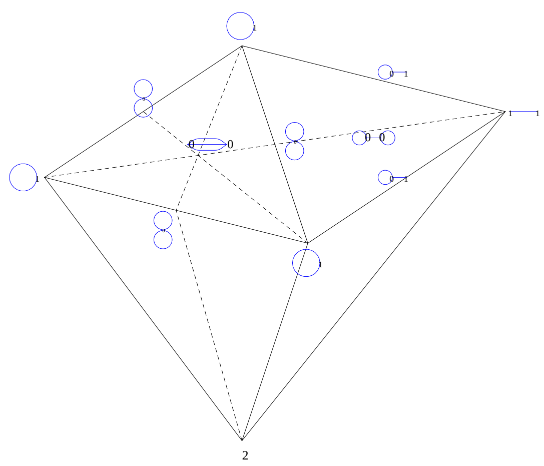


Figure 3: *The moduli space M_2^{tr} of genus 2 tropical curves. Each of the seven combinatorial types shown in Figure 1 gets a simplicial cone modulo a group action.*

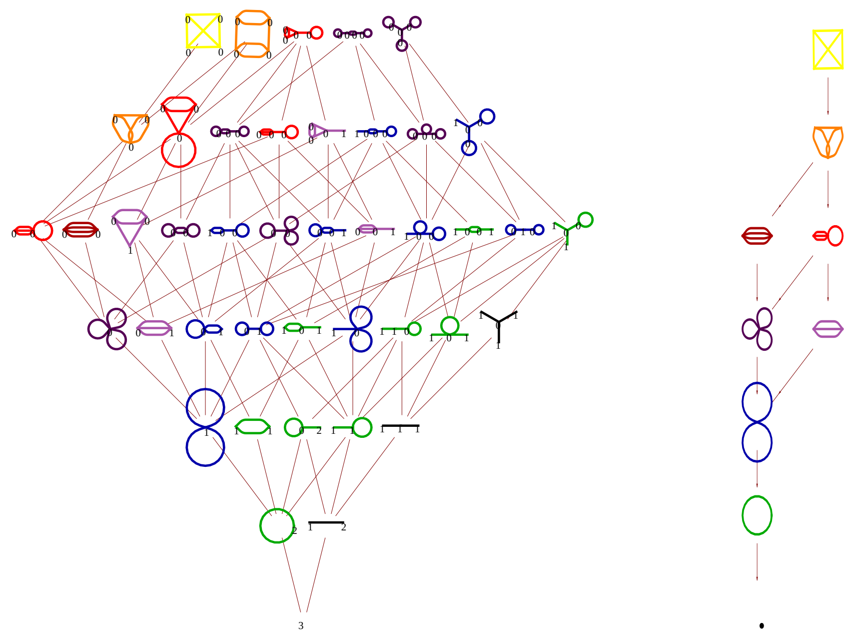


Figure 4: *Cells of the tropical moduli spaces M_3^{tr} and A_3^{tr}.*

(a) (b)

References

[1] S. Brannetti, M. Melo, F. Viviani, On the tropical Torelli map, Advances in Mathematics 226 (2011) 2546–2586.

[2] M. Chan, Combinatorics of the tropical Torelli map, arXiv:1012.4539.