On the semiampleness of the positive part of CKM Zariski decompositions

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Preliminaries(1)

Convention

X is a normal complex projective variety.
K_X is the canonical divisor on X.

We recall some definitions:

- A $\mathbb{Q}$-*divisor* $D = \sum a_i D_i$ is a $\mathbb{Q}$-linear combination of prime Weil divisors;
- Given a $\mathbb{Q}$-divisor $D = \sum a_i D_i$, the *round down* of D is

$$[D] = \sum [a_i] D_i.$$

- A $\mathbb{Q}$-divisor D is $\mathbb{Q}$-*Cartier* if there exists a multiple mD which is a Cartier divisor.

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- A $\mathbb{Q}$-Cartier divisor D is nef if $(D \cdot C) \geq 0$ for every irreducible curve $C \subseteq X$.
- A $\mathbb{Q}$-Cartier divisor D is big if there exists $C > 0$ such that

$$h^0(X, \mathcal{O}_X(mD)) \geq C \cdot m^{dim\, X},$$

for all sufficiently divisible $m \in \mathbb{N}$.

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Graded rings

Definition

Given a $\mathbb{Q}$-Cartier divisor D on X we can consider the graded ring

$$R(X, D) := \bigoplus_{mD \text{ Cartier}} H^0(X, \mathcal{O}_X(mD)),$$

where the multiplication is given by :

$$H^0(X, \mathcal{O}_X(mD)) \otimes H^0(X, \mathcal{O}_X(nD)) \longrightarrow H^0(X, \mathcal{O}_X((m+n)D))$$

- The graded ring of a $\mathbb{Q}$-Cartier divisor is not finitely generated in general as a $\mathbb{C}$-algebra.
- The finite generation of the graded ring associated to the (log) canonical divisor has been proved recently (Birkar, Cascini, Hacon, McKernan) and it has a fundamental role in the context of the Minimal Model Program.

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Zariski decomposition

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A big $\mathbb{Q}$-Cartier divisor D admits a Zariski decomposition in the sense of Cutkoski-Kawamata-Moriwaki (or a CKM-Zariski decomposition) if there exist two $\mathbb{Q}$-Cartier divisors P, N such that:

1. $D = P + N$;
2. P is nef and N is effective;
3. $H^0(X, \mathcal{O}_X([mP])) \simeq H^0(X, \mathcal{O}_X([mD]))$ for all $m \in \mathbb{N}$.

Remarks:

- Every big $\mathbb{Q}$-Cartier divisor on a surface admits a CKM-Zariski decomposition, in fact this is the classical Zariski decomposition;
- CKM-Zariski decompositions do not exist in general (Cutkoski's and Nakayama's counterexamples).

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Graded rings and Zariski decomposition

- Let D be a big $\mathbb{Q}$-Cartier divisor. If $D = P + N$ is a CKM-Zariski decomposition, then $R(X, D) \simeq R(X, P)$.
- Thus in order to check the finite generation of $R(X, D)$ we can study the finite generation of $R(X, P)$.
- This might be easier because P is nef. In particular $R(X, P) \simeq R(X, D)$ is finitely generated if and only if mP is base point free for some $m \in \mathbb{N}$.

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If P is a $\mathbb{Q}$-Cartier divisor such that, for some integer m, mP is Cartier and base point free, then P is said to be *semiample*.

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The starting point of our work is an important theorem by Kawamata. Let us begin with a less general version of it:

Theorem (Kawamata, 1987)

Let X be a smooth variety. If

1. K_X is the canonical divisor on X;
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In particular the canonical ring $R(X, K_X)$ is finitely generated.

In fact by Birkar, Cascini, Hacon, McKernan $R(X, K_X)$ is always finitely generated and a birational pullback of K_X admits a Zariski decomposition whenever K_X is big.

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Pairs

In the context of the Minimal Model Program instead of simply working with a variety, it is usual to work with pairs, given by a variety and a $\mathbb{Q}$-divisor on it.

The motivation is that the canonical divisor K_X is not a Cartier (or a $\mathbb{Q}$-Cartier) divisor in general.

This leads to the following definition:

Definition

Let X be a variety and let $\Delta = \sum a_i D_i$ be an effective $\mathbb{Q}$-divisor on X such that all $a_i \leq 1$. We say that (X, Δ) is a pair if $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

The idea is that we add a "small" divisor to K_X to make it $\mathbb{Q}$-Cartier.

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SNCS divisors and log-resolutions

Let $\Delta = \sum a_i D_i$ be a $\mathbb{Q}$-divisor on a smooth variety X. We say that Δ has *simple normal crossing support (SNCS)* if each D_i is smooth and the components of Δ intersect "as transversely as possible".

Thanks to Hironaka's theorem we can turn every divisor into a SNCS one, by performing a suitable birational morphism:

Definition

Let (X, Δ) be a pair. A *log resolution* of (X, Δ) is a birational morphism $\mu : X' \to X$ such that X' is smooth and $\mu^*(\Delta) + \mathrm{exc}(\mu)$ has SNCS. Here we denote by $\mathrm{exc}(\mu)$ the sum of all the exceptional divisors of μ.

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LC centers

Given a pair (X, Δ) we can consider some subvarieties of X that are special with respect to it:

Definition

A subvariety $V \subseteq X$ is a *LC center* of the pair (X, Δ) if there exists a log resolution $\mu : X' \to X$ such that if we write

$$K_{X'} - \mu^*(K_X + \Delta) = \sum a_E E$$

then there exists E, prime divisor on X', such that $\mu(E) = V$ and $a_E \leq -1$.
We say that V is a *pure* LC center if $a_E = -1$ for every such E.

Definition

A pair (X, Δ) is *Kawamata log terminal (KLT)* if it has no LC centers.
It is *log canonical (LC)* if it has only pure LC centers.

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Example 1: Take X to be smooth and suppose $\Delta = \sum a_i D_i$ has SNCS (and $a_i \leq 1$). Then $\mu = id : X \to X$ is a log resolution.
Note that

$$K_X - \mu^*(K_X + \Delta) = -\Delta = -\sum a_i D_i.$$

It follows that the LC centers of (X, Δ) are the divisors $\{D_i \ : \ a_i = 1\}$ and all the irreducible components of finite intersections of these divisors.
Hence (X, Δ) is KLT if and only if $a_i < 1$ for every i and (X, Δ) is always LC as long as all $a_i \leq 1$.

From this example we see that:

Remark

One reason for a subvariety $V \subseteq X$ to be an LC center of (X, Δ) is that V is a divisor with coefficient 1 in Δ, or an irreducible component of an intersection of divisors with coefficient 1.

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Then $\mu : X' = Bl_p X \to X$ is a log-resolution and it is easy to see that

$$K_{X'} - \mu^*(K_X) = (n - d)E,$$

where E is the exceptional divisor of μ, so that $\mu(E) = p$.
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Let X be a smooth variety. If D is a $\mathbb{Q}$-Cartier divisor such that

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then the positive part P is semiample.

Kawamata's theorem(2)

Now we can give a more general version of Kawamata's theorem, working in the context of pairs:

Theorem (Kawamata, 1987)

Let (X, Δ) be a KLT pair. If D is a $\mathbb{Q}$-Cartier divisor such that

1. $aD - (K_X + \Delta)$ is nef for some $a \geq 0$;
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then the positive part P is semiample.

In particular the graded ring $R(X, D)$ is finitely generated.

LC case

Question

What happens if we take (X, Δ) to be a LC pair? Does the result still hold?

Note that LC pairs are usually much more difficult to treat than KLT pairs.
In fact:

- KLTness is an open condition, it is maintained if we slightly perturb the divisor of the pair. LCness is not.
- If (X, Δ) is KLT than X has only rational singularities. The same is not true for LC pairs.

Answer

No, the result does not hold in general! But...

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No, the result does not hold in general! But...

Counterexample (Zariski-Mumford)

Let $C_0 \subseteq \mathbb{P}^2$ be a smooth cubic curve and let L be the hyperplane class on $\mathbb{P}^2$. Take 12 points $p_1, \ldots, p_{12}$ on C_0 such that $\mathcal{O}_{C_0}(p_1 + \cdots + p_{12} - 4L)$ is a non-torsion line bundle of degree zero on C_0.

Consider the blow-up along the 12 points:

$$\mu : X = Bl_{12}\mathbb{P}^2 \to \mathbb{P}^2$$

and denote by $E = \sum_{i=1}^{12} E_i$ the sum of the exceptional divisors. We put $D = 4\mu^*(L) + E$ and $\Delta = \widetilde{C_0} \sim 3\mu^*(L) - E = -K_X$. It is easy to see that

- The pair (X, Δ) is LC;
- D is big and nef ($D = P$ is a trivial CKM Zariski decomposition);
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Logbig divisors

In the previous example, the only LC center of the pair (X, Δ) is $V = Supp(\Delta)$.

The positive part of the Zariski decomposition $P = D$ is such that $H^0(mP_{|_V}, V) = 0$.

In other words, though asymptotically P has a lot of sections (it is big), it loses all its positivity when we restrict it to V.

We try to consider big divisors that "behave well" with respect to the LC centers of the given pair:

Definition (Miles Reid)

Let (X, Δ) be an LC pair and let P be a big $\mathbb{Q}$-divisor on X. Then P is said to be *logbig* with respect to the pair (X, Δ) if $P_{|_V}$ is big for every LC center $V \subseteq X$.

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Remark: The conjecture holds if

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Idea of the proof

We treat in a different way LC centers of different dimensions.
What we use is the following claim (in any dimension):

Claim

Under the usual assumptions we can prove semiampleness of P if

1. $P_{|_V}$ is big for every V divisorial LC center;
2. $P_{|_Z}$ is semiample, where $Z = \bigcup_{V \text{ LC center of codim.} \geq 2} V$.

If $dimX \leq 3$ and P is logbig then hypothesis 1. holds by logbigness.

Hypothesis 2. holds because every V of codim ≥ 2 is a curve or a point.
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We reduce our claim to the following:

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Under the usual assumptions P is semiample if we know that $Bs(|mP|) \cap Nklt(X, \Delta) = \emptyset$, where $Nklt(X, \Delta) = \bigcup_{V \text{ LC center}} V$

- We use the bigness of P when restricted to the divisorial LC centers to construct an "almost" LC pair (X, Δ') such that $aD - (K_X + \Delta')$ is ample.
- We consider $\mu : (Y, \Delta_Y) \to (X, \Delta')$ a log-resolution, so that Y is smooth and Δ_Y is SNCS;
- By the above ampleness we can slightly perturb Δ_Y so that all the divisorial LC centers of (Y, Δ_Y) are contracted by μ;
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Remarks

- We also proved the conjecture in any dimension if the pair (X, Δ) is divisorial log terminal (DLT);
- We proved a similar statement with the additional hypothesis that $(1-\epsilon)\Delta$ is KLT for some $\epsilon > 0$ and in some particular non-LC cases;
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Thank you!