Big q-Ample Divisors

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Main Result

A line bundle L on X is q-ample if for every coherent sheaf $\mathcal{F}$ on X, there exists an integer m_0 such that $m \geq m_0$ implies $H^i(X, \mathcal{F} \otimes L^{\otimes m}) = 0$ for $i > q$.
Totaro showed that this property is equivalent to other previously studied properties in characteristic 0, most notably the property of uniform q-amplitude studied by Demailly, Peternell and Schneider in [2]. In particular, the q-amplitude of a line bundle depends only on its numerical class, and the cone of such bundles is open.
This means that there is some hope of recovering geometric and numerical information about X and its subvarieties from knowing when a line bundle is q-ample, though at present such results are known only in limited cases. In general much is known about the 0-ample cone (which is the ample cone) and the $(n-1)$-ample cone of an n dimensional variety X is the negative of the complement of the pseudoeffective cone of X. For intermediate values of q the relation between numerical and cohomological data remains mysterious.
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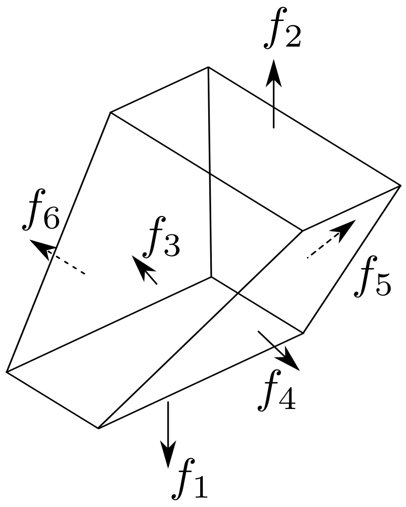
Theorem. *[1] Let X be complex projective scheme, and let L be a line bundle on X. Then L is q-ample on X if and only if the restriction of L to the augmented base locus of L is q-ample.*

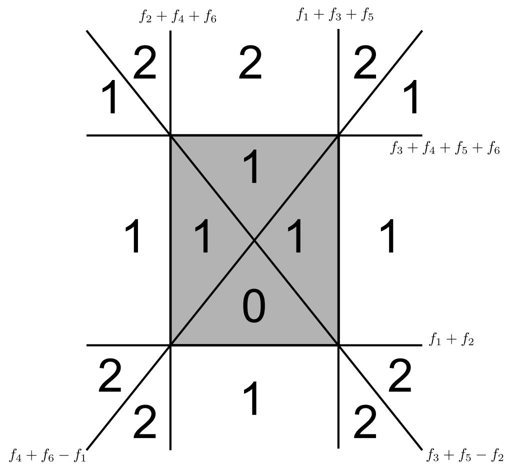
When $q = 0$, the Kleiman criterion says that we can check when a bundle is in the closure of the ample cone by checking its positivity on the curves of X. Using our theorem, we get an analogous criterion for when $q = n - 2$.

Corollary. *Let X be a nonsingular projective variety. A big line bundle L on X is $(n-2)$-ample iff the restriction of L^* to every irreducible codimension 1 subvariety is not pseudoeffective.*

Example

Totaro [3] has given an example of a smooth toric 3-fold with a line bundle L which is not in the closure of the 1-ample cone, but the restriction of L to every 2 dimensional subvariety is in the closure of the 1-ample cone of each subvariety.
Let X be the projectivization of the rank 2 vector bundle $\mathcal{O} \oplus \mathcal{O}(1,-1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Then X is a smooth toric Fano 3-fold, and the associated fan Σ has 6 rays, which we label $f_1 \dots f_6$. On the left is a picture of the dual polytope to Σ. We can think of a torus invariant divisor on X as an integral combination of the codimension 1 torus orbits. Denote by F_i the torus orbit corresponding to the ray f_i. The divisor F_i can be thought of as a piecewise linear function on $|\Sigma|$ which sends f_i to 1 and the other rays to 0. Two divisors are linearly equivalent iff their functions differ by a linear function.





The second figure shows a slice of $N^1(X) \otimes \mathbb{R}$, where the effective cone is shaded. The numbers in each region are the largest q such that a linebundle in the interior of that region is q-ample. Each dividing hyperplane is a minimal linear dependence among the rays f_i. One can show that $L' = 3F_1 + 3F_2 - F_3 - F_4 - F_5 - F_6$ is not 1-ample on X but is 1-ample when restricted to any of the F_i, and the same is true for any other divisor in the top middle region of the second figure.

References

[1] Morgan V Brown, *Big q-ample line bundles*, 2011, arXiv:1105.3449.

[2] Jean-Pierre Demailly, Thomas Peternell, and Michael Schneider, *Holomorphic line bundles with partially vanishing cohomology*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993) (Ramat Gan), Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., 1996, pp. 165–198. MR 1360502 (96k:14016)

[3] Burt Totaro, *Line bundles with partially vanishing cohomology*, J. Eur. Math. Soc.,to appear.