Diagonal Frobenius Splittings

Bonn International Graduate School in Mathematics

universität bonn

Achinger, Piotr

Advisor : Prof. Dr. Nicolas Perrin

Frobenius splittings

Let X be an algebraic variety over an algebraically closed field of characteristic $p > 0$. Let $F : X \to X$ be the **(absolute) Frobenius morphism** (i.e., the identity on the topological space $|X|$ and the map $F^* = (f \mapsto f^p) : \mathcal{O}_X \to F_*\mathcal{O}_X$ on the sheaf of functions).

Definition 1. *A* **Frobenius splitting** *of X is an $\mathcal{O}_X$-linear map $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_X$ such that $\sigma \circ F^* = id_{\mathcal{O}_X}$. If such a splitting exists, we say that X is* **Frobenius split**.

Simply put, a Frobenius splitting on $X = \operatorname{Spec} R$ is an additive map $\sigma : R \to R$ satisfying $\sigma(f^p g) = f\sigma(g)$.

This is an important notion since on one hand, Frobenius split varieties satisfy various **cohomological vanishing** results (which can be lifted to characteristic zero by semicontinuity), and on the other, such varieties are **ubiquitous** whenever one considers varieties with group action arising from representation theory.

Consequences 1. Assume that X is Frobenius split. Then

1. For any **ample** line bundle $\mathcal{L}$ on X, we have
$$H^i(X, \mathcal{L}) = 0 \quad \text{for} \quad i > 0.$$
2. X is **reduced** and **semi-normal**.
3. The **Kodaira vanishing theorem** ($H^i(X, \mathcal{L}^{-1}) = 0$ for $\mathcal{L}$ ample and $i < \dim X$) holds.

Definition 2. *A Frobenius splitting σ is said to be* **compatible** *with a closed subscheme $Y \subseteq X$ if $\sigma(F_*\mathcal{I}_Y) \subseteq \mathcal{I}_Y$ where $\mathcal{I}_Y$ is the sheaf of ideals of Y.*

Consequences 2. Assume that X is Frobenius split compatibly with a closed subscheme Y. Then

1. The given Frobenius splitting **induces a Frobenius splitting** of Y.
2. If $\mathcal{L}$ is an ample line bundle on X, the **restriction map** $H^0(X, \mathcal{L}) \to H^0(Y, \mathcal{L}|_Y)$ **is surjective**.
3. If the given splitting is compatible with Y', then it is also compatible with their **intesection** $Y \cap Y'$ (in particular, $Y \cap Y'$ **is reduced**).

Definition 3. *We say that X is* **diagonally Frobenius split** *if $X \times X$ is Frobenius split compatibly splitting the diagonal $\Delta_X \subseteq X \times X$.*

Consequences 3. Assume that X is diagonally Frobenius split. Then

1. If $\mathcal{L}$ and $\mathcal{L}'$ are ample line bundles on X, the **multiplication map** $H^0(X, \mathcal{L}) \otimes H^0(X, \mathcal{L}') \to H^0(X, \mathcal{L} \otimes \mathcal{L}')$ **is surjective**.
2. Every ample line bundle on X is **very ample**.
3. Any projective embedding of X is **projectively normal**.

Main question. Which varieties are diagonally Frobenius split?

Toric varieties

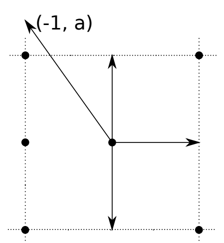
Every **toric variety** has a **unique equivariant Frobenius splitting** and this splitting compatibly splits all invariant subvarieties. In fact, the push-forward $F_*\mathcal{O}_X$ can be calculated as the following direct sum of line bundles:

Fact 1. Let $m(D)$ be the number of effective T-invariant divisors in $|D|$ with coefficients $< p$. Then
$$F_*\mathcal{O}_X = \bigoplus_{[D] \in \operatorname{Pic} X} \mathcal{O}_X(-D)^{m(pD)}.$$

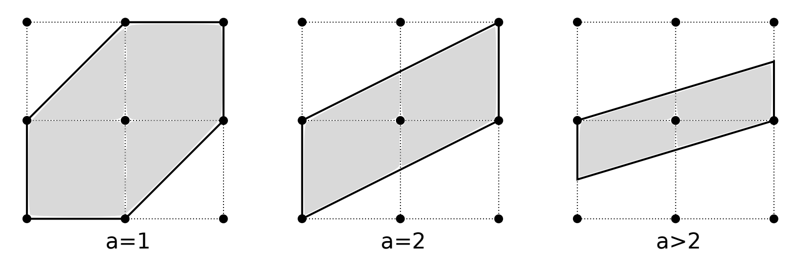
If we ask for *diagonal* Frobenius splittings, the question becomes more complicated.

Theorem 1 (Payne). *Let X be a toric variety defined by a fan Σ in a lattice N, let M be the dual lattice and let $\rho_1, \ldots, \rho_s \in N$ be the ray generators of Σ. Define the polytope*
$$\mathbb{F}_X = \{x \in M \otimes \mathbb{Q} : -1 \leq \langle \rho_i, x \rangle \leq 1\} \subseteq M \otimes \mathbb{Q}.$$
Then X is diagonally Frobenius split if and only if $(p-1)\mathbb{F}_X \cap M$ maps onto M/pM.

Example. For a positive integer a, the **a-th Hirzebruch surface** F_a is given by the following fan:



The polytopes $\mathbb{F}_X$ for $X = F_1, F_2, F_a$ $(a > 2)$ are pictured below:



We can see that

- F_1 is diagonally Frobenius split for all p,
- F_2 is diagonally Frobenius split for all odd p,
- F_r is not diagonally Frobenius split for $r > 2$.

Question 1. What are the possibilities for the set S of all primes p such that X is diagonally Frobenius split in characteristic p?

In all known examples, S either consists of **all primes**, consists of **all odd primes** or is **empty**.

A link with combinatorics

Trying to answer Question 1 one comes across the following result:

Fact 2. If $2 \in S$ then all primes are in S.

This turned out to have a surprising link with **hypergraph discrepancy** explained below.

Definition 4. *Let $\mathcal{K}$ be a* **hypergraph**, *i.e., a family of nonempty subsets of a finite set A.*

1. *Its* **discrepancy** *is the number*
$$\operatorname{disc} \mathcal{K} = \min_{f : A \to \{-1,1\}} \max_{S \in \mathcal{K}} \left| \sum_{i \in S} f(i) \right|.$$
2. *The* **hereditary discrepancy** *of $\mathcal{K}$ is defined as*
$$\operatorname{herdisc} \mathcal{K} = \max_{B \subseteq A} \operatorname{disc}(\mathcal{K}|_B)$$
where $\mathcal{K}|_B = \{S \cap B : S \in \mathcal{K}\}$ is the ,,induced hypergraph".
3. *By* **linear discrepancy** *of $\mathcal{K}$ we mean the number*
$$\operatorname{lindisc} \mathcal{K} = \max_{\alpha_1, \ldots, \alpha_n \in [0,1]} \min_{\varepsilon_1, \ldots, \varepsilon_n \in \{0,1\}} \max_{S \in \mathcal{K}} \left| \sum_{i \in S} (\alpha_i - \varepsilon_i) \right|.$$

Simply put, hereditary discrepancy says **how well one can color** arbitrary subsets B of A using two colors such that each hyperedge of $\mathcal{K}$ has roughly the same number of vertices in B of both colors. Linear discrepancy is a sort of **linear approximation** to this problem (and thus easier to compute).

The relationship between hereditary and linear hypergraph discrepancy was studied by Lovász, Spencer and Vesztergombi. They proved the following remarkable result:

Theorem 2. *For any hypergraph $\mathcal{K}$ we have* $\operatorname{lindisc} \mathcal{K} < \operatorname{herdisc} K$.

This result is the **key step** in proving Fact 1. Furthermore, there is a following correspondence:

(toric varieties which are diagonally Frobenius split in characteristic 2)
$\leftrightarrow$
(hypergraphs of hereditary discrepancy 1)
$\leftrightarrow$
(totally unimodular matrices).

A matrix is called **totally unimodular** if the determinants all its square submatrices are -1, 0 or 1.

Spherical varieties

Definition 5. *A* **spherical variety** *is a normal variety X on which a reductive group G acts in such a way that a Borel subgroup $B \leq G$ has a dense orbit on X.*

Thus, spherical varieties are a common generalization of **toric varieties** and **homogeneous spaces**. Since we understand diagonal Frobenius splitting for both of these classes, we might hope for a criterion for general spherical varieties.

The first test case is to consider **toroidal horospherical varieties**, which are fibrations over a homogeneous space $Z = G/P$ with fibers toric varieties Y. They are constructed in the following way: Let G be a semisimple simply connected algebraic group. Fix a Borel subgroup $B \leq G$ and let U be its unipotent part. Let H be an arbitrary subgroup of G containing U and let $P = N_G(H)$ be its normalizer. Then P is a parabolic subgroup and H is the intersection of kernels of characters of P. Thus, the fibration $G/H \to G/P = Z$ is a principal bundle for the torus $T' = P/H$. We can take an arbitrary toric variety Y on which the torus T' acts and form the associated bundle $X = G/H \times^{T'} Y$ which is fibered over $Z = G/P$ with fiber Y.

Question 2. Which toroidal horospherical varieties are diagonally Frobenius split?

Ideally, we would like to have a criterion similar to that for the toric variety Y. We have computed the space where splittings live:

Fact 3. We have
$$\operatorname{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X) = \bigoplus_{\lambda \in (p-1)\mathbb{F}_Y \cap X(T')} V(\lambda + 2(p-1)\rho_P),$$
where λ ranges over the characters of T' (which is a sublattice in $X(P)$) and $V(\lambda) = H^0(G/P, \mathcal{L}(\lambda)) = H^0(G/B, \mathcal{L}(\lambda))$ is the **dual Weyl module** and ρ_P is the sum of positive simple roots corresponding to P (then $\mathcal{L}(2\rho_P)$ is the canonical divisor of G/P).

References

1. Michel Brion, Shrawan Kumar *Frobenius Splitting Methods in Geometry and Representation Theory*, Birkhäuser 2005.
2. L. Lovasz, J. Spencer and K. Vesztergombi *Discrepancy of set systems and matrices*, European Journal of Combinatorics 7 (1986), 151160.
3. Sam Payne *Frobenius splittings of toric varieties*, Algebra and Number Theory 3 (2009), no. 1, 107–119.

2011 Hausdorff Center for Mathematics
piotr.achinger@hausdorff-center.uni-bonn.de