Convex and Projective Duality : Different Points of Views on the Theorems on Tangencies

Roland Abuaf

Under the direction of Laurent Manivel

Institut Fourier, Université Joseph Fourier, Grenoble, France.

Abstract

We discuss phenomena of tangency in Convex Optimization and Projective Geometry. Both theories have at disposal a powerful theory of duality. In both cases, the duality allows a nice interpretation of the contact locus of a hyperplane with an embedded variety. In this poster, we try to investigate more precisely the similarities between the theorems on tangencies existing in both theories. We focus in particular on a theorem of Anderson and Klee and its reformulation in the context of Algebraic Geometry, known as a conjecture of Ranestad and Sturmfels. If true, this conjecture would have significant consequences for Projective Geometry.

1 Introduction

Let $X \subset \mathbb{R}^n$ be a compact convex body whose interior contains 0. There have been considerable efforts to classify the *singularities* of the points lying in the boundary of X. A clear picture of the situation was probably given for the first time by Anderson and Klee [AK52].

Definition 1.0.1 *Let $X \subset \mathbb{R}^n$ be a compact convex body whose interior contains 0, let $X^* \subset \mathbb{R}^{n*}$ be its dual body and let $x \in \partial X$. We say that x is an* **r-singular point** *of X if the* **exposed face** *of X^* relative to $x^{\perp}$ has dimension at least r.*

Theorem 1.0.2 *Let $X \subset \mathbb{R}^n$ be a compact convex body whose interior contains 0 and let $r \in \{0, ..., n-1\}$. The set of* r-singular *points of X can be covered by countably many compact subsets of finite $n-r-1$-dimensional Hausdorff measure.*

Now we turn to a similar situation in Projective Geometry. Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be an irreducible, non-degenerate projective variety. Zak found a bound for the dimension of the contact locus of any linear space with X [Zak93].

Theorem 1.0.3 *Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be an irreducible, non degenerate projective variety and let $L \subset \mathbb{P}^N_{\mathbb{C}}$ be a linear space. Denote by $X_L = \{x \in X, T_{X,x} \subset L\}$, we have the inequality:*

$$\dim(X_L) \leq \dim(L) - \dim(X) + b + 1,$$

where $b = \dim X_{sing}$.

Both theorems tell us that *support loci* are subject to dimensional constraints. However, theorem 1.0.2 bounds the dimension of a family of hyperplanes when the dimension of the contact locus of the general member is known, whereas theorem 1.0.3 bounds the dimension of the contact locus of a single hyperplane. Theorem 1.0.2 has no proven analogue in Algebraic Geometry and the equivalent statement is known as a conjecture of Ranestad and Sturmfels [RS10a].

Conjecture 1.0.4 *Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a non-degenerate, irreducible projective variety and let $X^* \subset \mathbb{P}^N_{\mathbb{C}}$ be its projective dual. Let $r \in \{0, ..., n-1\}$ and denote by $X^*\langle r\rangle = \{H^{\perp} \in X^*, \dim\langle X_H\rangle \geq r\}$, where $\langle . \rangle$ denotes the scheme-theoretic linear span and X_H is the tangency locus of H with X. We have the inequality:*

$$\dim X^*\langle r\rangle \leq n - r - 1.$$

Most of the content exposed here (except proposition 2.2.4 and section 3) is well known, either from the analyst or the algebraic geometer. The idea for this presentation was inspired both by the body of work [RS10a], [RS10b] and discussions with Kristian Ranestad and Bernd Sturmfels. I am grateful to them for sharing their ideas with me. I would also like to thank the GAeL organizers for giving me the opportunity to attend the 2011 issue.

2 Dualities and Contact Loci

2.1 A Common Setting for the Dualities

Here we will formulate, in a common language, the duality for convex bodies and for projective varieties. In the following, the space $\mathbb{E}^n$ either denotes the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ or the real euclidean space $\mathbb{R}^n$. An object $X \subset \mathbb{E}^n$ refers to a compact convex body in $\mathbb{R}^n$ whose interior contains 0 or to a reduced (irreducible) projective scheme in $\mathbb{P}^n_{\mathbb{C}}$. If X is a convex body, then ∂X is the boundary of X. If X is a reduced projective scheme, then $\partial X = X$. We denote by $\overline{Z}$ the convex hull or the Zariski closure of an object $Z \subset \mathbb{E}^n$.

Definition 2.1.1 *Let $X \subset \mathbb{R}^n$ be a compact convex body whose interior contains 0, let $y \in X$ and let $H \subset \mathbb{R}^n$ be a hyperplane. We say that H has* **contact** *with X at y, if for all $x \in X$ we have $\langle H^{\perp}, x\rangle \leq 1$, and $\langle H^{\perp}, y\rangle = 1$, where $\langle ., . \rangle$ is the evaluation pairing between $\mathbb{R}^{n*}$ and $\mathbb{R}^n$.*

Note that if H has contact with X at y, then necessarily $y \in \partial X$.

Definition 2.1.2 *Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a reduced (irreducible) projective scheme and let $H \subset \mathbb{P}^n_{\mathbb{C}}$ be a hyperplane.*
Let $y \in X_{smooth}$. We say that H has **contact** *with X at y if $T_{X,y} \subset H$.*
Let $y \in X_{sing}$. We say that H has **contact** *with X at y if there exist sequences $(y_m) \in X_{smooth}$ and $(H_m{}^{\perp}) \in \mathbb{P}^{n*}_{\mathbb{C}}$ such that $H^{\perp} = \lim H_m^{\perp}$, $y = \lim y_m$ and H_m has contact with X at y_m for all $m \in \mathbb{N}$.*

Now we can state both dualities in a common setting.

Theorem 2.1.3 (Duality) *Let $X \subset \mathbb{E}^n$ be an object. Consider the incidence:*

$$I_X = \{(H^{\perp}, x) \in \mathbb{E}^{n*} \times \partial X, H \ \ has \ \ contact \ \ with \ \ X \ \ at \ \ x\},$$

and the natural diagram:

$$\begin{array}{ccc} & I_X & \\ {}^{q}\swarrow & & \searrow^{p} \\ \mathbb{E}^{n*} & & \partial X \subset \mathbb{E}^n \end{array}$$

Let $X^* = \overline{q(I_X)}$. We have $I_{X^*} = I_X$. As a consequence, we have $X^{**} = X$ and $q(I_X) = \partial(X^*)$.

Note that if $X \subset \mathbb{E}^n$ is a reduced projective scheme, then $q(I_X)$ is obviously Zariski closed. In this case, the equality $q(I_X) = \partial(X^*)$ is a bit meaningless since, in our notations, $\partial X^* = X^*$. Note also that, by construction, for all $H^{\perp} \in q(I_X)$, we have:

$$x \in p(q^{-1}(H^{\perp})) \Leftrightarrow H \ \ has \ \ contact \ \ with \ \ X \ \ at \ \ x.$$

The object $p(q^{-1}(H^{\perp}))$ is called the ***contact locus*** of H along X. In Convex Geometry, the set $p(q^{-1}(H^{\perp}))$ is often called the *exposed face* of X relative to H, while in Projective Geometry it is known as the *tangency locus* of H along X. The duality says that the set of hyperplanes which have contact with X at x is equal to the contact locus of $x^{\perp}$ along X^*. That is, for all $x \in X$, we have:

$$H^{\perp} \in p(q^{-1}(x)) \Leftrightarrow x^{\perp} \ \ has \ \ contact \ \ with \ \ X^* \ \ at \ \ H^{\perp}.$$

2.2 The Principle of Anderson and Klee

In this section, we formulate the principle of Anderson and Klee in a common setting for Projective Geometry and Convex Geometry.

Notations 2.2.1 *Let $X \subset \mathbb{E}^n$ be an object. The* **linear span** *of X, which we denote by $\langle X\rangle$ is the smallest linear subspace of $\mathbb{E}^n$ which contains X.*

In the case $Z \subset \mathbb{E}^n$ is a non-reduced scheme, the subspace $\langle Z\rangle$ is the scheme-theoretic linear span of Z.

Definition 2.2.2 *Let $X \subset \mathbb{E}^n$ be an object. A point $x \in X$ is said to be a* **r-singular point** *in X if $\dim\langle q(p^{-1}(x))\rangle \geq r$. The set of r-singular points of X is denoted by $X\langle r\rangle$.*

The following result is the archetype of the theorem on tangencies which should be true in all geometries. It was proven by Anderson and Klee (see [AK52], or [Sch93] for a modern presentation) in the context of Convex Geometry and it is known as a conjecture of Ranestad and Sturmfels [RS10a] in Projective Geometry.

Conjecture 2.2.3 *Let $X \subset \mathbb{E}^n$ be an object, we have the inequality:*

$$\dim X\langle r\rangle \leq n - r - 1.$$

Here the dimension must be understood as the Hausdorff dimension or the algebraic dimension, depending on the context.
Using the theory developped by Hironaka around the notion of normal flatness [Hir64] and a result of Lê-Teissier [LT88], one can prove the following statement in Projective Geometry.

Proposition 2.2.4 *Let $X \subset \mathbb{E}^n$ be an irreducible, reduced projective variety. Let*

$$X\langle r\rangle_{top} = \{x \in X, \dim|\langle q(p^{-1}(x))|_{red}\rangle \geq r\},$$

where $|Y|_{red}$ denotes the reduced space underlying the scheme Y. We have the inequality:

$$\dim X\langle r\rangle_{top} \leq n - r - 1.$$

3 Applications to Projective Geometry

If true, the conjecture of Ranestad and Sturmfels would have significant consequences for Projective Geometry. In fact, even proposition 2.2.4 can be used to prove a generalization of Severi's theorem.

Notations 3.0.5 *Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be an irreducible projective variety. We denote by $\widetilde{X}(r)$ the set $\widetilde{X}(r) = \{x \in X, \dim q(p^{-1}(x)) \geq r\}$.*

Theorem 3.0.6 *Let $X \subset \mathbb{P}^5_{\mathbb{C}}$ be a smooth, irreducible, non-degenerate projective surface and let $X^* \subset \mathbb{P}^{5*}$ its projective dual. We have $\dim \widetilde{X^*}(1) \leq 2$, with equality if and only if X is the Veronese surface.*

<u>*Sketch of the proof*</u> :

► By assumption $X \neq \mathbb{P}^2$, so X does not contain a 2-dimensional family of lines. As a consequence of proposition 2.2.4, we see that $\dim \widetilde{X^*}(1) \leq 2$.

Assume that $\dim \widetilde{X^*}(1) = 2$, proposition 2.2.4 again shows that for all $H^{\perp} \in \widetilde{X^*}(1)$, the curve-components of $(H \cap X)_{sing}$ are plane curves.

Let $H^{\perp} \in \widetilde{X^*}(1)$ be a general point and let k be the maximum of the degree of the curve-components of $|(H \cap X)_{sing}|_{red}$. Assume that $k \geq 3$. Then, there is a plane curve, say C, in $|(H \cap X)_{sing}|_{red}$ such that all lines in $\langle C\rangle$ are trisecants to X. But this is true for general $H^{\perp} \in \widetilde{X^*}(1)$, so that a careful count of dimension shows that we have a 4-dimensional family of trisecants to X. This is impossible by the trisecants lemma.

As a consequence, the smooth surface X is covered by a 2-dimensional family of conics, it is the Veronese surface.

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Note that Theorem 3.0.6 obviously implies Severi's original result. Indeed, if $X \subset \mathbb{P}^5_{\mathbb{C}}$ is a smooth, irreducible, non-degenerate surface whose secant variety does not cover the ambiant space, then Terracini's lemma implies that $\dim \widetilde{X^*}(1) = 2$. Another proof of Severi's result, relying on similar techniques as the above ones, is due to Zak and is a consequence of theorem 1.0.3. Hence, one may hope that theorem 1.0.3 and theorem 2.2.3 could be considered in a common setting. As such, these results are perhaps incarnations of a deeper principle, which has yet to be discovered.

References

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