Half way from point counting to integration

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Abstract

INTEGRATION is an important tool to study real and complex manifolds. For varieties over finite fields, counting rational points can be seen as a discrete analogue of integration. The theory of p-adic integration combines arithmetic and analytic aspects of these two ideas and we apply this to the study of complex geometry of certain moduli spaces and p-adic representation theory.

p-adic integration

LET p be any prime number and denote by $(\mathbb{Q}_p, |\cdot|)$ the field of p-adic numbers with its non-archimedian norm, which makes it into complete, locally compact space. Furthermore we denote by $\mathbb{Z}_p$ and $\mathfrak{m}$ the ring of p-adic integers resp. its maximal ideal. The residue field $\mathbb{Z}_p/\mathfrak{m}$ is isomorphic to the field with p elements $\mathbb{F}_p$.
Let $\mathcal{X}$ be a smooth scheme over $S = \text{Spec}(\mathbb{Z}_p)$, flat and of relative dimension n. We have a natural inclusions of rational points $\mathcal{X}(\mathbb{Z}_p) \subset \mathcal{X}(\mathbb{Q}_p)$. Since $\mathbb{Q}_p$ is complete and $\mathcal{X}$ smooth, $\mathcal{X}(\mathbb{Z}_p)$ and $\mathcal{X}(\mathbb{Q}_p)$ are in fact compact resp. locally compact analytic manifolds.
Similar as for real manifolds, we can integrate top dimensional differential forms on these spaces. Using the fact, that every unit in $\mathbb{Z}_p$ has norm 1, this gives a canonical measure $d\mu_{\mathcal{X}}$ on $\mathcal{X}(\mathbb{Z}_p)$. Since $\mathcal{X}(\mathbb{Z}_p)$ is compact, the so defined p-adic volume is finite and has a nice interpretation due to A. Weil.

Theorem. *(Weil, 1982, [1]) Let $\mathcal{X}$ be a smooth scheme over S of relative dimension n. Then*

$$\int_{\mathcal{X}(\mathbb{Z}_p)} d\mu_{\mathcal{X}} = \frac{|\mathcal{X}(\mathbb{F}_p)|}{p^n}.$$

One application of this theory was given by V. Batyrev in [2]. Using the Weil conjectures and the theorem above he proved

Theorem. *(Batyrev, 1999, [2]) Let X, Y be smooth projective varieties over the complex numbers with trivial canonical bundle. If X and Y are birational, then they have the same Betti numbers.*

My research

WE are interested in the following quotient construction. Let K be a field, G a reductive algebraic group over K and $\mathfrak{g}$ its Lie algebra. Consider a representation $\rho : G \to \text{GL}(V)$, where V is some finite dimensional K-vector space, and its derivative $\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$. Let G act on $M = V \times V^*$ via ρ with weight $(1, -1)$, and define the map $\mu : M \to \mathfrak{g}^*$ as

$$\langle \mu(v, w), X \rangle = \langle \varrho(X)v, w \rangle \,,$$

for $(v, w) \in M$ and $X \in \mathfrak{g}$.
For $\xi \in (\mathfrak{g}^*)^G$ we define the affine GIT quotient

$$M//_\xi G = \text{Spec}((\mathcal{O}_{\mu^{-1}(\xi)})^G).$$

When $K = \mathbb{C}$ many interesting spaces arise in this way, e.g. Hilbert Schemes of n-Points on $\mathbb{C}^2$ or Nakajima quiver varieties.
In [3] the number of rational points over $K = \mathbb{F}_q$ of theses varieties are computed and this leads to formulas for their Betti number over $\mathbb{C}$.
Finally for $K = \mathbb{Q}_p$, both V and $\mathfrak{g}$ are locally compact groups, hence they come with Haar measures dv resp. dx. If we fix an additive character $\Psi : \mathbb{Q}_p \to S^1 \subset \mathbb{C}$ with $\ker(\Psi) = \mathbb{Z}_p$ we have the following p-adic version of the main proposition 1 in [3].

Proposition. *Let $a_\varrho : \mathfrak{g} \to \mathbb{C}$ be given by*

$$a_\varrho(x) = \int_{V(\mathbb{Z}_p)} \mathbb{1}_{V(\mathbb{Z}_p)}(\varrho(x)v)dv.$$

If $\xi \in (\mathfrak{g}^*)^G$ is a regular value of μ, the p-adic volume of $M//_\xi G$ is

$$\int_{\mathfrak{g}} a_\varrho(x)\Psi(\langle x, \xi \rangle)dx. \tag{1}$$

The function a_ϱ contains representation theoretic information about G. Hence we can read (1) in two ways: On one hand it could give more information on $M//_\xi G$ as in the finite field case, on the other hand by using the geometry of $M//_\xi G$ we hope to gain new insight into the representation theory of G over $\mathbb{Q}_p$.

References

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