Automorphisms on OG$_{10}$ Acting Trivially on Cohomology

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Introduction

In the classification of compact Ricci-flat manifolds a fundamental role is played by the following type of manifolds:

Definition 0.1. A compact kähler manifold X is called an *irreducible holomorphic symplectic manifold* (short: an *IHS manifold*) if the following hold:

- X is simply connected.
- $H^{2,0}(X) = \mathbb{C}\sigma_X$, where σ_X is a nowhere degenerate holomorphic 2-form.

Important examples are:

- $K3$ surfaces,
- Hilbert schemes of points on K3 surfaces (*Beauville*),
- moduli spaces of stable sheaves on K3 surfaces (*Mukai*).

O'Grady constructed a new ten-dimensional example of IHS manifolds as a symplectic resolution of a moduli space of semistable sheaves on a $K3$:

Example 0.2. Let (S, H) be a polarised $K3$ satisfying $H^2 = 2$. Denote by M the relative compactified Jacobian of degree 4 over the linear system $|2H|$. Then M admits a symplectic resolution $\tilde{M}$ which is a new example of an IHS manifold.

Remark 0.3. Let X be an IHS, then $H^2(X, \mathbb{Z})$ admits a natural non-degenerate lattice structure. Together with its weight-two Hodge structure this is the most important invariant of X.

The Cohomological Representation

Automorphisms are a powerful tool to understand the geometry of IHS manifolds. We consider the following representation:

$$\nu\colon Aut(X) \to O(H^2(X, \mathbb{Z})).$$

Remark 0.4. For $K3$s and their Hilbert schemes, $\ker \nu = \emptyset$. Thus we can study their automorphisms using lattice theory.
Warning: There are IHS manifolds with nontrivial $\ker \nu$.

We have the following fundamental result:

Theorem 0.5 (Hassett$-$Tschinkel, [HT13])**.** *The kernel of ν is a deformation invariant of the manifold X.*

With Giovanni Mongardi we were able to prove:

Theorem 0.6 (Mongardi$-$W., [MW14])**.** *Let X be a manifold deformation equivalent to O'Grady's ten-dimensional example, then ν is injective.*

Idea of the proof

By Theorem 0.5 we may choose X to be $\tilde{M}$ as in Example 0.2, where S is a double sextic ramified along a very special sextic curve.
By definition, M admits a fibration

$$\pi\colon M \to |2H|.$$

Now, let φ be an automorphisms of $\tilde{M}$ in $\ker \nu$. We have the following easy observations:

- φ descends to M and
- the fibration $M \to |2H|$ is φ-equivariant, i.e. fibres are mapped to fibres and we have an induced action φ^* on the base $|2H|$.

For a general curve $C \in |2H|$ (which is of genus 5), the fibre $\pi^{-1}(C)$ is the Jacobian $Jac^4(C)$, which contains the Theta divisor of effective line bundles.

Lemma 0.7. *The relative Theta-divisor of $M \to |2H|$ is rigid, thus preserved by φ.*

Since φ maps each fibre $Jac^4(C)$ to $Jac^4(\varphi^*(C))$, the Torelli theorem for Jacobians implies that, in fact, we have an isomorphism of the curves $C \simeq \varphi^* C$.

Proposition 0.8. *The rational map $|2H| \dashrightarrow \bar{M}_5$ is injective.*

Thus φ^* is trivial and we conclude that φ is trivial itself by again using the Torelli theorem for Jacobians.

References

[HT13] B. Hassett, Y. Tschinkel, *Hodge theory and lagrangian planes on generalized Kummer fourfolds,* Mosc. Math. J. vol. 13 (2013), no. 1 33–56.

[MW14] G. Mongardi, M. Wandel, *Induced Automorphisms on Irreducible Symplectic Manifolds,* Preprint, arXiv:1405.5706.