Cox rings and universal torsors over arbitrary fields of characteristic 0

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Origin

The close relation between Cox rings and universal torsors for toric varieties is well known since the work of Colliot-Thélène and Sansuc (1970s) and Cox (1990s).

$$\begin{array}{ccc} \mathbb{A}_k^{n-1} \setminus \{0\} & \subseteq & \operatorname{Spec} k[x_0, \dots, x_n] \\ \downarrow & & \\ \mathbb{P}^n_k & & \end{array}$$

More recently, it has been established for locally factorial varieties over algebraically closed fields making use of the additional notion of Cox sheaf [1]. Galois descent of universal torsors is well known. What about Cox rings?

Goal

To define Cox rings and Cox sheaves over non-closed fields, to study their existence and classification, and the relation to universal torsors.

Setting

k field of char 0, X geometrically integral variety over k, $\bar{k}[X]^\times = \bar{k}^\times$, $\operatorname{Pic}(X_{\bar{k}})$ finitely generated.

Definitions

over $\bar{k}$

- A *Cox ring* of $X_{\bar{k}}$ is a $\operatorname{Pic}(X_{\bar{k}})$-graded $\bar{k}$-algebra structure on $\oplus_{[D]\in\operatorname{Pic}(X_{\bar{k}})} H^0(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}(D))$ compatible with the map div.
- A *Cox sheaf* of $X_{\bar{k}}$ is a $\operatorname{Pic}(X_{\bar{k}})$-graded $\mathcal{O}_{X_{\bar{k}}}$-algebra structure on $\oplus_{[D]\in\operatorname{Pic}(X_{\bar{k}})} \mathcal{O}_{X_{\bar{k}}}(D)$.

over k

- A *Cox ring* of X is a k-algebra R such that $R \otimes_k \bar{k}$ is a Cox ring of $X_{\bar{k}}$ with compatible Galois action.
- A *Cox sheaf* of X is a $\mathcal{O}_X$-algebra $\mathcal{R}$ such that $\mathcal{R} \otimes_k \bar{k}$ is a Cox sheaf of $X_{\bar{k}}$ with compatible Galois action.

Construction 1

Let $\Lambda \subseteq \operatorname{CaDiv}(X_{\bar{k}})$ finitely generated and free such that the sequence $0 \to \Lambda_0 \to \Lambda \to \operatorname{Pic}(X_{\bar{k}}) \to 0$ is exact. Let $\chi : \Lambda_0 \to \bar{k}(X_{\bar{k}})^\times$ such that $\operatorname{div}(\chi(E)) = E$, $\forall\ E \in \Lambda_0$. Then

$$\left(\oplus_{D\in\Lambda} \mathcal{O}_{X_{\bar{k}}}(D)\right) / \left(1 - \chi(E) : E \in \Lambda_0\right)$$

is a Cox sheaf of $X_{\bar{k}}$.

Uniqueness over $\bar{k}$

Up to isomorphism, there is exactly one Cox sheaf and exactly one Cox ring of $X_{\bar{k}}$.

Automorphisms

- The group of automorphisms of a Cox sheaf of $X_{\bar{k}}$ is isomorphic to $\mathrm{H}_X(\bar{k})$.
- The group of automorphisms of a Cox ring of $X_{\bar{k}}$ is isomorphic to $\mathrm{H}_{X,\mathrm{eff}}(\bar{k})$.

Construction 2

Assume that the exact sequence

$$1 \to \bar{k}^\times \to \bar{k}(X)^\times \to \bar{k}(X)^\times/\bar{k}^\times \to 1 \qquad (1)$$

has a Galois equivariant splitting $\sigma : \bar{k}(X)^\times \to \bar{k}^\times$. Let $\mathcal{D} \subseteq \operatorname{CaDiv}(X_{\bar{k}})$ be a system of representatives for $\operatorname{Pic}(X_{\bar{k}})$. Define multiplication of homogeneous sections s_i of degree $D_i \in \mathcal{D}$ by $s_1 \cdot s_2 = s_1 s_2 f$, where $f \in \bar{k}(X)^\times$ is the unique element such that $D_1 + D_2 + \operatorname{div}(f) \in \mathcal{D}$ and $\sigma(f) = 1$. Then the sheaf

$$\oplus_{D\in\mathcal{D}} \mathcal{O}_{X_{\bar{k}}}(D)$$

is a Cox sheaf of $X_{\bar{k}}$ with a continuous Galois action.

Existence over k

- [2] For X smooth, universal torsors of X exist if and only if (1) has a Galois equivariant splitting.
- If (1) has a Galois equivariant splitting, then Cox sheaves and Cox rings of X exist.
- If $X(k) \neq \emptyset$, then Cox sheaves, Cox rings and universal torsors of X exist.

Classification over k

$$\begin{array}{ccccc}
H^1(k, \mathrm{H}_X) & & \left\{\begin{array}{c}\text{Cox sheaves of } X_{\bar{k}} \\ \text{with continuous} \\ \text{Galois action,} \\ \text{up to isomorphism}\end{array}\right\} & \xrightarrow{\mathcal{R} \mapsto \mathcal{R}(X)} & \left\{\begin{array}{c}\text{Cox rings of } X_{\bar{k}} \\ \text{with continuous} \\ \text{Galois action,} \\ \text{up to isomorphism}\end{array}\right\} \\
\updownarrow \text{1:1} & & \uparrow \text{1:1} \quad \begin{array}{c}\mathcal{R} \otimes_k \bar{k} \\ \uparrow \\ \mathcal{R}\end{array} & & \text{1:1} \uparrow \quad \begin{array}{c}R \otimes_k \bar{k} \\ \uparrow \\ R\end{array} \\
\left\{\begin{array}{c}\text{Universal torsors of } X, \\ \text{up to isomorphism}\end{array}\right\} & \xleftarrow[\operatorname{Spec}_X \mathcal{R} \leftarrow \mathcal{R}]{1:1} & \left\{\begin{array}{c}\text{Cox sheaves of } X, \\ \text{up to isomorphism}\end{array}\right\} & \xrightarrow{\mathcal{R} \mapsto \mathcal{R}(X)} & \left\{\begin{array}{c}\text{Cox rings of } X, \\ \text{up to isomorphism}\end{array}\right\}
\end{array}$$

If $\operatorname{Pic}(X_{\bar{k}})$ is generated by effective divisors, then all the horizontal arrows are bijective.

$\operatorname{Cl}(X)$ or $\operatorname{Pic}(X)$?

Construction 1 can be performed starting with Weil divisors [1] or line bundles [3] instead of Cartier divisors. If X is normal, and $\mathcal{R}_C, \mathcal{R}_W, \mathcal{R}_L$ are given by Construction 1 with Cartier divisors, Weil divisors and line bundles respectively, then

$$\mathcal{R}_C \subseteq \mathcal{R}_W \quad \text{and} \quad \mathcal{R}_C \cong \mathcal{R}_L.$$

Mori dream spaces

Let $\mathcal{R}$ be a Cox sheaf of X. If $\mathcal{R}(X) \otimes_k \bar{k}$ is finitely generated, then

$$\begin{array}{ccc} \operatorname{Spec}_X \mathcal{R} & \subseteq & \operatorname{Spec} \mathcal{R}(X) \\ \downarrow & & \\ X & & \end{array}$$

Smooth projective varieties over $\bar{k}$ with finitely generated Cox ring are called Mori dream spaces [4].

References

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