Singular irreducible symplectic varieties

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Irreducible symplectic varieties

Irreducible symplectic varieties are defined as compact holomorphically symplectic Kähler varieties with trivial fundamental group, whose symplectic structure is unique up to proportionality.

Very few deformation classes of irreducible symplectic varieties are known. The problem of extending the very short list of known deformation classes of irreducible symplectic varieties is very hard. One can get many more deformation classes in extending the scope to possibly singular irreducible symplectic varieties. Moreover, the singular irreducible symplectic varieties are a natural object of study, since they arise as moduli spaces of semistable sheaves (or twisted sheaves, or objects of the derived category endowed with a Bridgeland stability condition) on a K3 or abelian surface. Another important way to get singular irreducible symplectic varieties is to quotient smooth ones by symplectic automorphism groups. Moreover, as follows from Namikawa (see [5]), their deformation theory and period mapping behave similarly to the case of smooth ones.

My research

Among my research interests
The first examples of singular irreducible symplectic varieties appeared in [1], simultaneously with nonsingular ones, as finite quotients of products of two K3 surfaces or of 4-dimensional compact complex tori. In [2], Markushevich and Tikhomirov provided the first example of singular irreducible symplectic variety which is not obtained as a quotient of a smooth one. We call this variety the Markuchevitch–Tikhomirov variety and we denote it by $\mathcal{P}$.

An important tool for the study of irreducible symplectic varieties is the Beauville–Bogomolov form. Therefore, following the idea to develop the theory of singular irreducible symplectic varieties, the question of calculating the Beauville–Bogomolov form of the Markushevich–Tikhomirov varieties appears quite natural. I find the following Beauville-Bogomolov lattice: $U(2)^3 \oplus E_8(-1) \oplus (-2)^2$ (see [3]).

This result is obtained from the fact that $\mathcal{P}$ is related by a Mukai flop to a partial desingularisation of

$$S^{[2]}/(involution),$$

where S is a K3 surface and $S^{[2]}$ is the Hilbert scheme of two points.

I develop in my thesis many tools to calculate integer cohomology of a variety X quotiented by an automorphism group of prime order G (see [4]). These tools allowed us to calculate the Beauville–Bogomolov lattice of the partial desingularisation of $S^{[2]}/(involution)$, and then the Beauville–Bogomolov lattice of $\mathcal{P}$.

By the same technique, I calculate the Beauville–Bogomolov lattice of the quotient of an irreducible symplectic variety of $K3^{[2]}$-type by a numerically standard symplectic automorphism of order 3. We denote this 4-dimensional singular irreducible symplectic variety by M_3. Other examples of use of the technique of [4] are the calculation of the cup-product lattice of a K3 surface quotiented by a symplectic involution, by a symplectic and by a non-symplectic automorphism of order 3. We denote these quotients respectively by $\overline{Y_2}$, $\overline{Y_3}$ and $\overline{Z_3}$. I also calculate the cup-product lattice of a complex torus of dimension 2 quotiented by $-\mathrm{id}$; we denote this quotient by $\overline{A}$. We summarize the results of our calculations in the following table:

X/G	$H^2(X/G,\mathbb{Z})$
$\overline{Y}_2$	$E_8(-1) \oplus U(2)^3$
$\overline{Y}_3$	$U(3) \oplus U^2 \oplus A_2^2$
$\overline{Z_3}$	$U \oplus E_6$
$\overline{A}$	$U(2)$
M_3	$U(3) \oplus U^2 \oplus A_2^2 \oplus (-6)$
$\mathcal{P}$	$U(2)^3 \oplus E_8(-1) \oplus (-2)^2$.

Here $H^2(X/G,\mathbb{Z})$ is endowed with the cup-product for the surfaces and with the Beauville–Bogomolov form for the fourfolds.

References

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