Fundamental Groups and specialization maps

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Introduction

The étale fundamental group was exhaustively studied by Grothendieck in SGA1. In particular he proved that, if X is a proper scheme over S, then one can define the specialization map, and he studied the cases in which it is in fact an isomorphism.

Studying Galois and Tannakian categories one encounters other fundamental groups and the analogue of the specialization maps for these groups are not always known.

Étale specialization map

The results Grothendieck proved in [SGA I] about the specialization map are the following.

Theorem.
Let Y a locally Noetherian scheme, $f\colon X \to Y$ a proper morphism with geometrically connected fibres, y_0 and y_1 two points of Y such that $y_0 \in \overline{\{y_1\}}$, X_0 and X_1 the geometric fibres of X corresponding to the algebraic closures of $k(y_0)$ and $k(y_1)$, ξ_0 and ξ_1 geometric points of X_0 and X_1.
Then there exists a natural specialization homomorphism

$$\pi_1(X_1, \xi_1) \to \pi_1(X_0, \xi_0)$$

defined up to inner automorphisms.

If the morphism $f\colon X \to Y$ is also smooth he showed that

- If char $k(y_0) = 0$ then the specialization homomorphism is an isomorphism;
- If char $k(y_0) = p > 0$ then the specialization homomorphism induces an isomorphism between the maximal prime to p quotients

$$\pi_1(X_1, \xi_1)^{(p)} \to \pi_1(X_0, \xi_0)^{(p)}.$$

Proétale fundamental group

Grothendieck defines the coverings of a scheme X as finite étale morphisms over it. One would like to weaken this condition and consider a larger set of coverings.
In [BS13] the authors define the proétale fundamental group by introducing the notion of weakly étale morphisms.

A map of schemes $f\colon X \to Y$ is called *weakly étale* if both f and the diagonal morphism $\Delta_f\colon Y \to Y \times_X Y$ are flat.

The *proétale site* $X_{\text{proét}}$ is then defined as the category of weakly étale X-schemes, endowed with the fpqc topology.

Theorem. Let X be a locally topologically Noetherian connected scheme, ξ a geometric point of X, then the category Loc_X of locally constant sheaves over the proétale site is equivalent to the category of $\underline{\mathrm{Aut}(\xi^\star)\text{-}\textit{Sets}}$.

The *proétale fundamental group* is defined as $\pi^{\text{proét}}(X, \xi) := \mathrm{Aut}(\xi^\star)$.

Moreover, the pro-finite completion of the proétale fundamental group is the étale fundamental group, while its pro-discrete completion is the enlarged fundamental group defined in [SGA 3].

Tannakian fundamental group

The constructions of both the étale and the proétal fundamental groups are based on the notion of Galois categories. However, one can also consider Tannakian categories, and associate to them a group scheme.

A *neutral Tannakian category* is an abelian k-linear rigid tensor category $\mathcal{C}$ with k-bilinear tensor product, such that there exists a functor $F\colon \mathcal{C} \to \underline{k\text{-}Vec}$, which is called *fibre functor*.

Theorem. Let $\mathcal{C}$ be a neutral Tannakian category with fibre functor F, then there exists a group scheme G such that F induces an equivalence of categories between $\mathcal{C}$ and the category of representations *Rep*$_k(G)$.

If $\mathcal{C}$ is a neutral Tannakian category, then the associated group scheme G is called the *Tannakian fundamental group* of $\mathcal{C}$ via F.

Let X be a smooth scheme over k, an algebraically closed field of positive characteristic $p > 0$, then a *stratified bundle* on X is the data of a sequence of bundles E_i, and $\mathcal{O}_X$-linear isomorphisms $\sigma_i\colon {F_X}^\star E_{i+1} \cong E_i$, where F_X is the absolute Frobenius on X.

Given $\xi \in X$ a geometric point of X, then the category of stratified bundles is a Tannakian category, whose fibre functor is given by $F(\{E_i\}) = \xi^\star E_0$.

The group scheme associated to Str_X is called the *Tannakian fundamental group* $\pi_1^{Str}(X, \xi)$.

Questions

Let k be an algebraically closed field of positive characteristic, A a complete integrally closed Noetherian local ring, with residue field k, and denote $S = \mathrm{Spec}(A)$.

Suppose X is a proper scheme over S with geometrically connected fibres, let X_K denote the generic fibre, X_0 the closed fibre and fix ξ_0 and ξ_K geometric points.

- is it possible to define a specialization map $\pi_1^{Str}(X_K, \xi_K) \to \pi^{\text{proét}}(X_0, \xi_0)$?
- Are there examples in which is natural to consider the enlarged fundamental group, or Deligne's fundamental group instead of the proétale?

Bibliography

(SGA) A. Grothendieck, *Revêtements étales et groupe fondamental (SGA 1).*

(SGA3) M. Demazure, A.Grothendieck et al., *Schéma en groupes (SGA 3).*

(BS) B. Bhatt and P. Scholze, *The proétale topology for schemes.*