Do Derived Equivalent Varieties Have Equal Zeta Functions?

(Probably yes, at least sometimes)

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The derived category and Fourier–Mukai transforms

The star of the show is $D^b(X)$, the bounded derived category of coherent sheaves of a (smooth, projective) variety X. Its ways are still somewhat mysterious, and the question of what $D^b(X)$ tells us about X is a topic of active inquiry.

Any exact equivalence $D^b(X) \cong D^b(Y)$ of smooth, projective varieties is given by a Fourier–Mukai transform whose kernel is determined uniquely up to isomorphism [8].

Given varieties X and Y, a functor

$$F : D^b(X) \to D^b(Y)$$

is a Fourier–Mukai transform if there exists an object $P \in D^b(X \times Y)$, called a Fourier–Mukai kernel, such that

$$F \cong p_{2*}(p_1^*(-) \otimes P), \qquad \begin{tikzcd} & X \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

where pushforward, pullback, and tensor are all in their derived versions.

Examples

Many interesting examples of Fourier–Mukai equivalences are between the derived category of a variety and that of a fine moduli space, with the kernel given by the univeral sheaf of the moduli problem.

- Any abelian variety and its dual are derived equivalent via a Fourier–Mukai equivalence with kernel the Poincaré bundle.
- Any variety derived equivalent to a K3 surface can be shown to be another K3 surface that is a connected component of a moduli space classifying locally semi-stable sheaves (characteristic 0 case: [6, 4]; characteristic p case: [5]).

The derived category as an invariant

Preserved under derived equivalence (for instance):

- Dimension
- Canonical rings, hence
  - Kodaira dimension
  - Order of the canonical bundle
  - The variety itself, if it has ample or anti-ample canonical bundle; that is, in this case, derived equivalent varieties are isomorphic.

Not preserved under derived equivalence:

- Brauer groups
- π_1

Pairs of derived equivalent Calabi-Yau 3–folds that don't share these invariants have been found: see [1] and [2, 10], respectively.

The big picture

A conjecture of Orlov [9] states that, for smooth projective varieties:

Derived Equivalence $\Rightarrow$ Isomorphism of Motives

The search is on for evidence of this conjecture. A direction to pursue in the hunt: the conjecture predicts that derived equivalent varieties (over finite fields) should have equal zeta functions, but do they? Derived equivalence of varieties over fields of positive characteristic has not been investigated as much as over characteristic 0 fields.

Zeta functions

The zeta function of a nonsingular projective variety X over a finite field $\mathbb{F}_q$ is the following power series in t:

$$\zeta(X) = \exp\Big(\sum_{m\geq 1} N_m(X)\frac{t^m}{m}\Big),$$

where $N_m(X)$ is the number of points on X with coordinates in $\mathbb{F}_{q^m}$.

In [3], I answer the question of whether zeta functions are preserved under derived equivalence positively for a couple of large classes of varieties: abelian varieties and surfaces.

Abelian varieties

Let A and B be abelian varieties.

The key ingredient: If $D^b(A) \cong D^b(B)$, then $A \times \hat{A} \cong B \times \hat{B}$ [7, Theorem 2.19].

One can then argue that A and B must be isogenous, and hence, if they're over a finite field, they must have equal zeta functions (see, e.g. [11, Theorem 1]).

Surfaces

Strategy: Compare $\zeta(X)$ and $\zeta(Y)$ by comparing the l-adic étale cohomology groups of X and Y using this consequence of the Lefschetz fixed point theorem:

$$N_m(X) = (\Gamma_{\varphi^m} \cdot \triangle) = \sum (-1)^i \operatorname{Tr}(\varphi^{m*}|H^i(X, \mathbb{Q}_\ell))$$

where φ is the Frobenius map.

A Fourier–Mukai transform induces the following map on cohomology:

$$\Phi_P^H = p_{2*}(v(P) \cup p_1^*(-))$$

where, for any object $\mathcal{E}$ in a derived category $D^b(X)$, its Mukai vector is $v(\mathcal{E}) := \mathrm{ch}(\mathcal{E})\sqrt{\mathrm{td}(X)}$. If the transform is an equivalence, this map is an isomorphism.

The following diagram commutes

$$\begin{tikzcd} D^b(X) \arrow[r, "\Phi_P"] \arrow[d, "v"'] & D^b(Y) \arrow[d, "v"] \\ \tilde{H}^\bullet(X) \arrow[r, "\Phi_P^H"] & \tilde{H}^\bullet(Y) \end{tikzcd}$$

for each of the following Mukai–Hodge structures, respectively:

$$\widetilde{H}^{\mathrm{even}}(X) = \textstyle\bigoplus_{i=0}^{d_X} H^{2i}(X, \mathbb{Q}_\ell)(i),$$
$$\widetilde{H}^{\mathrm{odd}}(X) = \textstyle\bigoplus_{i=1}^{d_X} H^{2i-1}(X, \mathbb{Q}_\ell)(i).$$

We then have the equalities

$$\operatorname{Tr}(\varphi^*|\widetilde{H}^{\mathrm{even}}(X)) = \operatorname{Tr}(\varphi^*|\widetilde{H}^{\mathrm{even}}(Y)),$$
$$\operatorname{Tr}(\varphi^*|\widetilde{H}^{\mathrm{odd}}(X)) = \operatorname{Tr}(\varphi^*|\widetilde{H}^{\mathrm{odd}}(Y)).$$

Notice, however, that the Tate twists present mean that all the eigenvalues of these structures under the action of φ^* have the same weight, so there is not enough information here to distinguish between eigenvalues of different cohomology groups of the same parity.

The symmetries among cohomology groups are enough to show that $\zeta(X) = \zeta(Y)$ if X and Y are surfaces. The method can be generalized upward if enough of the cohomology groups vanish – for instance for 3–dimensional varieties with vanishing first cohomology group, including Calabi–Yau 3–folds.

No examples of derived equivalent varieties with unequal zeta functions are known at the time of the writing of this text.

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