The C_1 conjecture for hypersurfaces of toric varieties

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First

THE C_1 conjecture has already been proved for almost every known type of C_1 field :

- For finite fields $\mathbb{F}_q$, by H. Esnault in 2003.
- For fields of functions $\overline{k}(C)$ of a curve C over an algebraically closed field $\overline{k}$, by T. Graber, J. Harris, A.J. de Jong and J. Starr in 2003.
- For fields of fractions of formal series with coefficients in an algebraically closed field $\overline{k}((t))$.

It remains open essentially for the maximal unramified extension of the p-adic numbers $\mathbb{Q}_p^{nr}$.

All the results cited above have been proved using techniques depending strongly on the particular properties of the type of field they were dealing with.
An orthogonal approach to the conjecture has been initiated by J. Kollár in [1] : instead of considering **every** separably rationally connected variety over **a given** C_1 field, one can prove the conjecture for **a given** type of separably rationnally connected varieties, but over an arbitrary C_1 field admitting normic forms of any degree, hence over **every known** C_1 field.

Theorem (Th. 6.7 p.232 [1]). *Let K be a C_1 field admitting normic forms of arbitrary degree. Let $H_1, \ldots, H_r$ be hypersurfaces of the weighted projective space $\mathbb{P}_K(a_0, \ldots, a_n)$. If*

$$\sum_{j=1}^{r} \mathrm{wdeg}(H_j) < \sum_{i=0}^{n} a_i,$$

where wdeg denotes the weighted degree, then the intersection $H_1 \cap \cdots \cap H_r$ has a rational point over K.

This draws a notion of **low weighted degree** that is a sufficient condition for the existence of rational points over C_1 fields in intersections of hypersurfaces of weighted projective spaces.

Note that for a smooth hypersurface D of the weighted projective space $\mathbb{P}_{\mathbb{Q}_p^{nr}}(a_0, \ldots, a_n)$, we have

D has low weighted degree $\Leftrightarrow$ D is Fano

$\Leftrightarrow$ D is separably rationally connected

My research

KOLLÁR'S thoerem can be adapted to the case of a hypersurface D in a simplicial projective toric variety X_Σ (with split dense torus). This involves

- The **homogeneous equation** of D in the Cox ring of X_Σ and its **Newton polytope**.
- The birational geometry of the ambient variety X_Σ, i.e. tools from the **toric MMP**.

This defines (geometrically) the notion of **low toric degree** for D as a sufficient condition for the existence of rational points over C_1 fields.

Then it is sufficient to show that any smooth and separably rationally connected hypersurface of X_Σ has low toric degree to prove the C_1 conjecture for D (and for complete intersections of ample divisors in X_Σ) :

Theorem. *Let X_Σ be a simplicial projective toric variety defined over a C_1 field K such that K admits normic forms of arbitrary degree and the dense torus of X_Σ is split over K. Let D be a hypersurface of X_Σ defined over K. If D is smooth and separably rationally connected then it has low toric degree and in particular admits a rational point over K.*

Arising questions :

- Are hypersurfaces like D very special ? Are there general methods to embed nicely a (say Fano) variety in a well chosen toric ambient ?
- What happens to complete intersections of arbitrary divisors ? Is it possible to treat this way any subvariety ?

Others research interests :

- The **non quasiprojective** toric varieties, their Mori cone and their links to the projective world.
- **Mori Dream Spaces** and their "neat embeddings" in toric varieties.

References

[1] János Kollár, *Rational curves on algebraic varieties*, Springer, 1996.

[2] Olivier Wittenberg, *La connexité rationnelle en arithmétique*, Variétés rationnellement connexes: aspects géométriques et arithmétiques (Panor. Synthèses, 31, Soc. Math. France, Paris), 2010, pp. 61–114.