Bond theory for pentapods and hexapods

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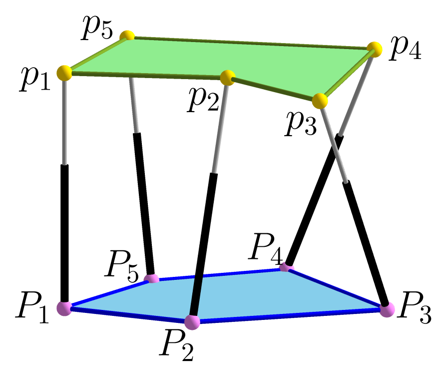
Introduction

We focus on mechanical manipulators called n–***pods***. Their geometry is defined by:

- n **base points** $P_i \in \mathbb{R}^3$ and n **platform points** $p_i \in \mathbb{R}^3$;
- n rigid bodies, called **legs**, connecting pairs of points (p_i, P_i), whose distance is d_i; at p_i and at P_i we have **spherical joints**.



An example of an n–pod for $n = 5$, namely a pentapod: base and platform are given by rigid bodies, and each of the five base points P_i is connected by a leg to exactly one platform point p_i. The joints at P_i and p_i allow only rotations.

Goal: given an n–pod L, **describe** the **direct isometries** σ of $\mathbb{R}^3$ satisfying

$$\|\sigma(p_i) - P_i\| = \|p_i - P_i\| \stackrel{\text{def}}{=} d_i \qquad \text{for all } i \in \{1, \dots, n\} \tag{1}$$

A New Compactification of SE_3

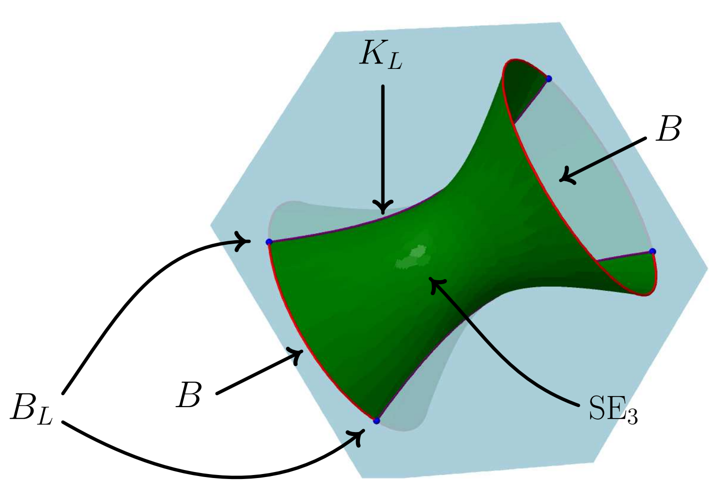
Fact: to study problems from kinematics, it is useful to **introduce compactifications into projective spaces** of SE_3, the group of direct isometries of $\mathbb{R}^3$.

Definition:
We introduce a compactification of SE_3 in $\mathbb{P}^{16}_{\mathbb{C}}$ and we call it X. We denote by B the ***boundary*** of X, namely the set $X \setminus \mathrm{SE}_3$.

Lemma:
Equation (1) **becomes linear** in the new coordinates of X.

Definition:
Given an n–pod L we define:

i. The ***configuration set*** K_L
$$K_L = \{\sigma \in \mathrm{SE}_3 \text{ satisfying Equation (1) for all } (p_i, P_i)\} \subseteq X$$
ii. The ***mobility*** of L, which equals the dimension of K_L.
iii. The set B_L of ***bonds***, $B_L = \overline{K_L} \cap B$, where $\bar{\cdot}$ denotes the Zariski closure.



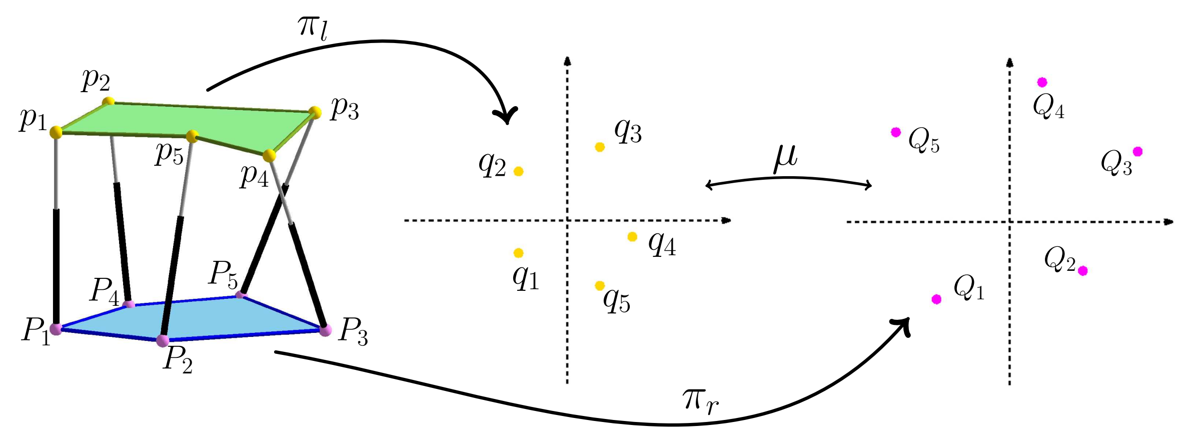
Schematic representation of the variety X: it is the disjoint union of SE_3 (green) and the boundary B (red); for any fixed n–pod L, Equation (1) defines n hyperplanes (light blue) cutting SE_3 in the configuration set K_L (purple) and intersecting B in the set B_L of bonds (blue dots).

Geometric Interpretation of Bonds

We distinguish four kinds of bonds: ***inversion***, ***similarity***, ***butterfly*** and ***collinearity*** bonds. The presence of each of these bonds implies **geometric conditions** on base and platform points.

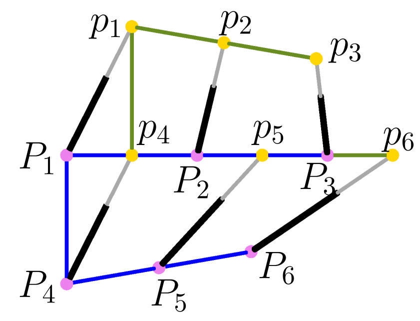
Figure 1 and Figure 4 are courtesly provided by Georg Nawratil. Figure 2 was created using Sage (http://www.sagemath.org) and Jmol (http://www.jmol.org).

Inversion/Similarity Bonds



We say that an n–pod admits an inversion/similarity bond if there exist directions $l, r \in S^2$ such that, if π_l, π_r are the orthogonal projections along l, r and $q_i = \pi_l(p_i)$, $Q_i = \pi_r(P_i)$ are the projections of base and platform points, then the q_i and the Q_i differ by an inversion/similarity μ. In view of the next section, notice that μ is a Möbius transformation if we identify the plane $\mathbb{R}^2$ with $\mathbb{C}$.

Butterfly/Collinearity Bonds



We say that an n–pod admits a butterfly bond if there exist lines g and G in $\mathbb{R}^3$ and an integer $m \leq n$ such that $P_1, \dots, P_m$ are aligned along g and $p_{m+1}, \dots, p_n$ are aligned along G.

Collinearity bonds are a limit case of butterfly bonds, when all base or platform points are collinear, or both.

Möbius Photogrammetry

Traditional Photogrammetry	Möbius Photogrammetry
Recover a finite set of **points** in $\mathbb{R}^3$ **from** finitely many **central projections**.	**Recover** (**up to similarity**) a finite set of **points** in $\mathbb{R}^3$ **from** finitely many **orthogonal projections**, known up to Möbius transformations.

Definition:
We denote by M_5 the moduli space of 5 points in $\mathbb{P}^1_{\mathbb{C}}$. Once we fix a vector $\vec{A} = (A_1, \dots, A_5)$ of points in $\mathbb{R}^2$ we can define a ***photographic map***:
$$f_{\vec{A}} : S^2 \longrightarrow M_5$$
sending $\varepsilon \in S^2$ to the equivalence class in M_5 of $\pi_\varepsilon(\vec{A})$, where π_ε is the orthogonal projection along ε to a plane $\mathbb{R}^2 \cong \mathbb{C} \hookrightarrow \mathbb{P}^1_{\mathbb{C}}$.

Position of $\vec{A}$	Behavior of $f_{\vec{A}} : S^2 \longrightarrow M_5$
Not coplanar	**Birational** to a rational curve of deg. 10 or 8
Coplanar, but not collinear	$2:1$ to a rational curve of deg. $5, 4, 3$, or 2

One can prove that the set of images under the **photographic map can be used to determine**, up to similarities, the vector of **points** we started with, as shown by the following result:

Theorem:
Let $\vec{A}$ and $\vec{B}$ be two 5–tuples of points in $\mathbb{R}^3$ such that no 4 points are collinear. Assume that $f_{\vec{A}}(S^2) = f_{\vec{B}}(S^2)$. If $\vec{A}$ is **coplanar**, then $\vec{B}$ is also **coplanar** and **affine equivalent** to $\vec{A}$. If $\vec{A}$ is **not coplanar**, then $\vec{B}$ is **similar** to $\vec{A}$.

Corollary:
If the mobility of a pentapod is 2 or higher, then one of the following conditions holds:

(a) Platform and base points are **similar**.

(b) Platform and base points are **planar** and **affine equivalent**.

(c) There exists $m \leq 5$ such that the points $p_1, \dots, p_m$ are **collinear** and the points $P_{m+1}, \dots, P_5$ **coincide**.

Reference: Matteo Gallet, Josef Schicho and Georg Nawratil. *Bond theory for pentapods and hexapods*. Submitted, available at http://arxiv.org/abs/1404.2149.