Problems about rationality of K3 and del Pezzo surfaces

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K3 surfaces

Let S be a surface. We say that S is a *K3 surface* if its canonical divisor K_S is trivial and $\dim H^1(S, \mathcal{O}_S) = 0$. Examples of K3 surfaces are smooth quartic surfaces in $\mathbb{P}^3$. Little is known about the arithmetic of K3 surfaces.

Question 1. *Is there a K3 surface S defined over $\mathbb{Q}$ or any number field such that its set of rational points is neither empty nor Zariski dense?*

In [3], Logan, McKinnon, and van Luijk prove the Zariski density of rational points for a large family of K3 surfaces. In [2], using similar techniques, we prove the following theorem.

Theorem 2. *Let c_1, c_2 be two nonzero rationals and S be the surface defined as*

$$S\colon x^4 - 4c_1^2 y^4 - c_2 z^4 - 4c_2 w^4 = 0.$$

Let $P = (x_0 : y_0 : z_0 : w_0)$ be a rational point on S with x_0 and y_0 both nonzero.
If $|2c_1|$ is a square in $\mathbb{Q}^\times$, then also assume that z_0, w_0 are not both zero. Then the set of rational points on the surface is Zariski dense.

Let S be a K3 surface defined over a number field k. We say that the rational points on S are *potentially dense* if there exists a finite extension k'/k such that the set $S(k')$ of k'-rational points is Zariski dense. In [1] Bogomolov and Tschinkel prove the following theorem.

Theorem 3. *Let S be a K3 surface defined over a number field k. Assume that S has a structure of an elliptic fibration or an infinite group of automorphisms. Then the rational points on S are potentially dense.*

Question 4. *Let S be a K3 surface defined over a number field k. Assume that S does not admit an elliptic fibration and its group of automorphisms is finite. Are the rational points of S potentially dense?*

Del Pezzo surfaces

We say that a variety X of dimension n over a field k is *unirational* if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$, defined over K. A *del Pezzo surface* is a smooth, projective, geometrically integral variety X of which the anticanonical divisor $-K_X$ is ample. We define the *degree* of a del Pezzo surface X as the self intersection number of K_X, that is, $\deg X = K_X^2$.
In [4] it is proven that many del Pezzo surfaces defined over a field k having a rational point are unirational. Building on work by Manin (see [4]), C. Salgado, D. Testa, and A. Várilly-Alvarado prove that all del Pezzo surfaces of degree 2 over a finite field are unirational as well, except possibly for three isomorphism classes of surfaces (see [5]). Recently, I and Ronald van Luijk proved that these remaining three cases are also unirational, thus proving the following theorem.

Theorem 5. *Every del Pezzo surface of degree 2 over a finite field is unirational.*

Question 6. *What about del Pezzo surfaces of degree 2 over infinite fields and del Pezzo surfaces of degree 1 over any field?*

References

[1] F. A. Bogomolov and A. Tschinkel, *Density of rational points on elliptic K3 surfaces*, Asian J. Math. **4** (2000), no. 2, 351–368.

[2] D. Festi, *Density of rational points on a family of diagonal quartic surfaces*, Master thesis, Universiteit Leiden.

[3] A. Logan, D. McKinnon, and R. Van Luijk, *Density of rational points on diagonal quartic surfaces*, Algebra and Number Theory **4(1)** (2008), 1–20.

[4] Yu. I. Manin, *Cubic forms: algebra, geometry, arithmetic*, North-Holland publishing, 1986.

[5] Cecília Salgado, Damiano Testa, and Anthony Várilly-Alvarado, *On the unirationality of del Pezzo surfaces of degree two*, preprint (2013).