On k-jet ampleness of line bundles on hyperelliptic surfaces

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Introduction

First let us recall the definition and basic properties of hyperelliptic surfaces.

Definition. A hyperelliptic surface S, sometimes called bielliptic, is a surface with $\kappa(S) = 0$ and $q(S) = 1$, where $\kappa(S)$ denotes the Kodaira dimension of S and $q(S) = \dim H^1(X, \mathcal{O}_X)$ is the irregularity of S.

Alternatively, a surface S is hyperelliptic if $S \cong (A \times B)/G$, where A and B are elliptic curves, and G is an Abelian group acting on A by translation and acting on B, such that A/G is an elliptic curve and $B/G \cong \mathbb{P}^1$; G acts on $A \times B$ coordinatewise.

We have the following situation:

$$\begin{array}{ccc} S \cong (A \times B)/G & \xrightarrow{\Phi} & A/G \\ \Psi \big\downarrow & & \\ B/G \cong \mathbb{P}^1 & & \end{array}$$

where Φ and Ψ are natural projections.

General fibers of Φ (respectively Ψ) are isomorphic to B (respectively to A). We denote them by B and A in the group $\mathrm{Num}(S)$ of divisors on S modulo numerical equivalence.

General fibres of Φ are smooth because the morphism $A \to A/G$ is finite and unramified (it is a morphism of the group A into the group A/G). The fibre of Ψ over a point is numerically equivalent to a multiple of a smooth elliptic curve, whose multiplicity equals the number of points in the fibre over x by the morphism $B \to B/G \cong \mathbb{P}^1$.

Hyperelliptic surfaces were classified at the beginning of XX century by G. Bagnera and M. de Franchis in [1]. They showed that there are seven non-isomorphic types of hyperelliptic surfaces. These types are characterised by the action of G on the elliptic curve B. Namely, if $B \cong \mathbb{C}/(\mathbb{Z}\omega \oplus \mathbb{Z})$, where $\omega \in \mathbb{C}$, $-\frac{1}{2} \le \mathrm{Re}\,\omega < \frac{1}{2}$, $\mathrm{Im}\,\omega > 0$ and $|\omega| \ge 1$, if $\mathrm{Re}\,\omega \le 0$; $|\omega| > 1$, if $\mathrm{Re}\,\omega > 0$, we have the theorem

Theorem (Bagnera-de Franchis)**.** Let $S \cong (A \times B)/G$ be a hyperelliptic surface, let $B \cong \mathbb{C}/(\mathbb{Z}\omega \oplus \mathbb{Z})$ where ω is as described above, and let $\rho = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$. Then the group G acts on A by translation, and the action of G on B is one of the following:

Type of a hyperelliptic surface	ω	G	Action of G on B
1	any	$\mathbb{Z}_2$	$x \mapsto -x$
2	any	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$x \mapsto -x$, $x \mapsto x + \varepsilon$, where $2\varepsilon = 0$
3	i	$\mathbb{Z}_4$	$x \mapsto ix$
4	i	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$x \mapsto ix$, $x \mapsto x + \frac{1+i}{2}$
5	ρ	$\mathbb{Z}_3$	$x \mapsto \rho x$
6	ρ	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$x \mapsto \rho x$, $x \mapsto x + \frac{1-\rho}{3}$
7	ρ	$\mathbb{Z}_6$	$x \mapsto -\rho x$

In 1990 F. Serrano in [4] characterised the group $\mathrm{Num}(S)$ for each of the surface's type:

Theorem (Serrano)**.** A basis of the group of classes of numerically equivalent divisors $\mathrm{Num}(S)$ for each of the surface's type and the multiplicities of the singular fibres in each case are the following:

Type of a hyperelliptic surface	G	$m_1, \ldots, m_s$	Basis of $\mathrm{Num}(S)$
1	$\mathbb{Z}_2$	$2, 2, 2, 2$	$A/2, B$
2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$2, 2, 2, 2$	$A/2, B/2$
3	$\mathbb{Z}_4$	$2, 4, 4$	$A/4, B$
4	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$2, 4, 4$	$A/4, B/2$
5	$\mathbb{Z}_3$	$3, 3, 3$	$A/3, B$
6	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$3, 3, 3$	$A/3, B/3$
7	$\mathbb{Z}_6$	$2, 3, 6$	$A/6, B$

Let $\mu = \mathrm{lcm}\{m_1, \ldots, m_s\}$ and let $\gamma = |G|$. For a hyperelliptic surface of any type, a basis of $\mathrm{Num}(S)$ consists of divisors A/μ and $(\mu/\gamma)\,B$.

We say that L is a line bundle of type (a, b) on a hyperelliptic surface if $L \equiv a \cdot (A/\mu) + b \cdot (\mu/\gamma)B$. In $\mathrm{Num}(S)$ we have that $A^2 = 0$, $B^2 = 0$, $AB = \gamma$, the last equality is a consequence of the fact that a hyperelliptic surface has a covering by a product of elliptic curves. Therefore if L_1 is of type (α_1, β_1), and if L_2 is of type (α_2, β_2) then $L_1.L_2 = \alpha_1\beta_2 + \beta_1\alpha_2$.

Let us finish this section with recalling the definition of k-jet ampleness of a line bundle.

Definition. A line bundle L is k-jet ample, if for each points $x_1, \ldots, x_r$ the morphism $H^0(X, L) \longrightarrow H^0\left(X, L \otimes \mathcal{O}_X/(m_{x_1}^{k_1} \otimes \ldots \otimes m_{x_r}^{k_r})\right)$ is surjective, where $\sum_{i=1}^{r} k_i = k + 1$.

Main result

We have obtained the following

Theorem. *Let S be a hyperelliptic surface of any type. Let L be a line bundle of type $(1, 1)$ on S. Then the line bundle $(k + 2)L$ is k-jet ample on S.*

Sketch of the proof. We prove the theorem using vanishing theorems: Kawamata-Viehweg theorem and Norimatsu lemma.

To show surjectivity of the morphism in the definition of k-jet ampleness it is enough to prove, by the long exact sequence, the vanishing of the cohomology group $H^1\left((K_S + L) \otimes m_{x_1}^{k_1} \otimes \ldots \otimes m_{x_r}^{k_r}\right)$. This, by the projection formula, is equivalent to showing that $H^1\left(K_{\widetilde{S}} + \pi^* L - \sum_{i=1}^{r}(k_i + 1)E_i\right) = 0$.

The main idea is to divide the proof in several cases depending on the sum of multiplicities k_i of points on the singular fibres. We consider whether this sum exceeds the number $\frac{k+1}{2}$ or not.

For example, for a hyperelliptic surface of type 1, whose basis of the $\mathrm{Num}(S)$ is generated by $A/2$ and B, we consider six cases:

I. On each fibre $A/2$, on each fibre A, and on each fibre B there are points with the sum of multiplicities k_i not greater than $\frac{k+1}{2}$.

II. There exists a fibre $A/2$ with sum of k_i greater than $\frac{k+1}{2}$. We have two subcases:

a. On each fibre B the sum of multiplicities k_i is smaller than $\frac{k+1}{2}$.

b. There exists a fibre B for which the sum of multiplicities k_i is equal at least to $\frac{k+1}{2}$.

III. There exists a fibre A with sum of k_i greater than $\frac{k+1}{2}$. Analogously, we have two subcases:

a. On each fibre B the sum of multiplicities k_i is smaller than $\frac{k+1}{2}$.

b. There exists a fibre B for which the sum of multiplicities k_i equals at least $\frac{k+1}{2}$.

IV. There exists a fibre B with sum of k_i greater than $\frac{k+1}{2}$, moreover on each fibre $A/2$ and on each fibre A the sum of multiplicities k_i is not greater than $\frac{k+1}{2}$.

We prove the vanishing of the cohomology group $H^1\left(K_{\widetilde{S}} + \pi^* L - \sum_{i=1}^{r}(k_i + 1)E_i\right)$ in each case separately, using Kawamata-Viehweg theorem in cases I and IIIa, and Norimatsu lemma in the remaining cases.

For hyperelliptic surfaces of type greater than 1, the number of cases may be different but the idea of the proof is the same.

The most difficult part is to show that $M\widetilde{C} \ge 0$ or $M\widetilde{C} > 0$, where M is a line bundle whose bigness and nefness (to get the assertion from Kawamata-Viehweg theorem) or ampleness (to conclude using Norimatsu lemma) we want to show, and $\widetilde{C}$ is the proper transform of an irreducible curve C, not numerically equivalent to any of the fibres, passing through points $x_1, \ldots, x_r$ with multiplicities $m_1, \ldots, m_r$.

To get the assertion we introduce an auxiliary divisor D and consider what we get from intersecting D with irreducible curves of type (α, β). □

Analogous theorem were obtained on Abelian and K3 surfaces (see [2], [3]), but the method we propose is different from methods used by other authors.

References

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