ABELIAN SURFACES AND HODGE THEORY

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Surfaces isogenous to a product

A SMOOTH surface S is said to be isogenous to a higher product of curves if it is isomorphic to a quotient $\frac{C_1 \times C_2}{G}$ where C_1 and C_2 are curves of genus greater or equal than 2 and G is a finite group acting freely on $C_1 \times C_2$ (see [2]). Consider surfaces isogenous to a higher product with $p_g = 1$ and $q = 0$. These have been studied and classified in [3]. The Hodge diamond is fixed:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 4 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Is it true that the second cohomology group $H^2(S, \mathbb{Q})$ decomposes, as rational Hodge structure, into a product of two rational structure of weight 1?
Equivalently, is it true that $H^2(S, \mathbb{Q}) \simeq H^2(E_1 \times E_2, \mathbb{Q})$ where E_1 and E_2 are elliptic curves?
Looking at the irreducible rational representations of G and its actions on the cohomology of the curves (following the ideas of [6] and [5]) we deduce that this holds for almost all examples. However some special cases appear and they seem to be related with elliptic curves with complex multiplication.
Further developments: Apply the theory developed by now to new surfaces, for example different families of surfaces isogenous to a product.

QM-abelian surfaces

A COMPLEX abelian surface A is a projective complex torus of dimension 2. An abelian surface A determines a $\mathbb{Q}$-algebra $End_{\mathbb{Q}}(A)$ defined by $End_{\mathbb{Q}}(A) = End(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. An abelian surface A has quaternion multiplication (or is of QM-type) if $End_{\mathbb{Q}}(A)$ is isomorphic to an indefinite quaternion algebra. It is well known that abelian surfaces of this kind are simple if the quaternion algebra is a skew-field (see [1]) i.e. they are not isogenous to a product of elliptic curves.
For a fixed indefinite quaternion algebra B consider the family of abelian surfaces A with $End_{\mathbb{Q}}(A) = B$. These families are well-known and are related with Shimura curves.
A classical example is [4] where the family related to the algebra $B_6 = \mathbb{Q}(-6, 2)$ is studied:

$$B_6 = \mathbb{Q} + \mathbb{Q}\mathbf{i} + \mathbb{Q}\mathbf{j} + \mathbb{Q}\mathbf{k}, \qquad \mathbf{i}^2 = -6,\ \mathbf{j}^2 = 2,\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

For some quaternion algebras B the elements of the family admit an automorphism of order 3 or 4 wich fix a polarization of type $(1, d)$. Under this condition we want to construct a projective model of the Shimura curve associated to the family.
Further developments: We focused on a special case with an automorphism of order 3 fixing a polarization of type $(1, 2)$. For other polarizations there is still some work to be done.

References

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