Categorical Lattice Conjecture : some examples.

Anthony Blanc
Postdoc at the Faculty of Mathematics of the
University of Vienna, Austria.

Categorical Geometry

LET X be a complex algebraic variety.

Idea : *Study X via the abelian category $QCoh(X)$ of quasi-coherent sheaves on X (Gabriel).*

Homotopical idea : *Study X via the differential graded category $L_{\mathrm{perf}}(X)$ of perfect complexes of quasi-coherent sheaves on X (Grothendieck, Orlov, Kontsevitch,...).*
The association

$$X \longmapsto L_{\mathrm{perf}}(X)$$

provides a functor

$$Var_{\mathbb{C}}^{op} \longrightarrow dgCat_{\mathbb{C}}.$$

Some material available for varieties is available for dg-categories :

Algebraic Varieties X	dg-Categories T (e.g. $T = L_{\mathrm{perf}}(X)$)
de Rham cohomology $\mathrm{H}^*_{\mathrm{DR}}(X, \mathbb{C})$	Periodic cyclic homology $\mathrm{HP}_*(T\vert\mathbb{C})$
Algebraic K-theory $K_*(X)$	Algebraic K-theory $K_*(T)$
Chern map $K_*(X) \longrightarrow \mathrm{H}^*_{\mathrm{DR}}(X, \mathbb{C})$	Chern map $K_*(T) \longrightarrow \mathrm{HP}_*(T\vert\mathbb{C})$

Such that if in the right column we set $T = L_{\mathrm{perf}}(X)$ we recover the left column.

Motto of Categorical Geometry : *Generalise theorems known for algebraic varieties to some larger class of dg-categories.*

Question :

Hodge structures on the $\mathrm{H}^*_{\mathrm{DR}}(X, \mathbb{C})$'s	Noncommutative Hodge structure on $\mathrm{HP}_*(T\vert\mathbb{C})$?

Refined question :

Hodge filtration on Ω^*_X or degeneration of the Hodge-to-de-Rham spectral sequence	Degeneration of $\mathrm{HH}_*(T\vert\mathbb{C})((u)) \Rightarrow \mathrm{HP}_*(T\vert\mathbb{C})$ (known in the case T is a bounded above dg-algebra by Kaledin)
Integral lattice $\mathrm{H}^*_{\mathrm{B}}(X, \mathbb{Z}) \longrightarrow \mathrm{H}^*_{\mathrm{DR}}(X, \mathbb{C})$	What's the categorical counterpart ?

The Lattice Conjecture

IN my thesis I defined the topological K-theory of $\mathbb{C}$-dg-categories $K^{\mathrm{top}}_*(T)$ endowed with a Chern map, which gives a proposal of answer. The topological K-theory verifies :

- $K^{\mathrm{top}}_* : dgCat_{\mathbb{C}} \longrightarrow GrAb$ is a localizing invariant in the sense of Cisinski–Tabuada.
- $K^{\mathrm{top}}_*(\mathrm{Spec}(\mathbb{C})) = \mathbb{Z}[\beta]$ with $deg(\beta) = 2$.
- If X is a smooth complex algebraic variety, then $K^{\mathrm{top}}_*(L_{\mathrm{perf}}(X)) = K^*_{\mathrm{top}}(X(\mathbb{C}))$.

Conjecture. *For every smooth and proper dg-category T over $\mathbb{C}$, the map $K^{\mathrm{top}}_*(T) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \mathrm{HP}_*(T|\mathbb{C})$ is an isomorphism.*

Theorem (B.)**.** *Let B be an associative unital $\mathbb{C}$-algebra which is finite dimensional over $\mathbb{C}$. Then the map $Ch^{\mathrm{top}} \otimes \mathbb{C} : K^{\mathrm{top}}_*(B) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \mathrm{HP}_*(B|\mathbb{C})$ is an isomorphism. Moreover we have the formula*

$$\mathrm{HP}_i(B|\mathbb{C}) \simeq colim_{i\geq 0}\pi_i(|\mathrm{Vect}^B|^{ST}) \otimes_{\mathbb{Z}} \mathbb{C},$$

where $|\mathrm{Vect}^B|^{ST}$ is the stabilization of the topological realization of the stack of compact right B-modules.

Strategy of proof :

- The result is true for semi-simple algebras because these are Morita equivalent to a product of copies of $\mathbb{C}$.
- Reduce to the semi-simple case doing the usual induction on the degree of nilpotency of the radical.
- Proving the K-theoretic analog of Goodwillie's result that HP is invariant under infinitesimal extensions.

Work in progress : The case of a stacky quotient $T = L_{\mathrm{perf}}([X/G])$ with X a smooth G-variety, G finite.

Strategy : Vistoli's formula for G-equivariant algebraic K-theory

$$K([X/G]) \simeq \prod_{g\in\mathcal{S}} (K(X^g) \wedge_{\mathbb{S}} R < g >)^{hZ_g}$$

allows us to reduce to the case of smooth varieties X^g.

Future investigation : The case of matrix factorization 2-periodic dg-categories $T = MF(X, f)$ with X a smooth variety and $f : X \longrightarrow \mathbf{A}^1$ regular.

Strategy :

- Adapt the construction of K^{top}_* for 2-periodic dg-categories.
- Use Efimov's statement that $\mathrm{HP}_*(MF(X, f))$ is equivalent as a bundle with connection on the formal punctered disk to $\mathrm{H}^{*-1}_{an}(f^{-1}(0), \phi_X \mathbb{C}_X)$, with connection given by monodromy twisted by sign.