On Real Rationally Connected Varieties

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Preliminary Definitions

Let $X/\mathbb{C}$ be a smooth projective variety. We say X is **rationally connected** if it satisfies any of the following equivalent definitions

- For every $x_1, x_2 \in X$ there is a morphism $f : \mathbb{P}^1 \to X$ satisfying $x_1, x_2 \in f(\mathbb{P}^1)$.
- For every $x_1, \ldots, x_n \in X$ there is a morphism $f : \mathbb{P}^1 \to X$ satisfying $x_1, \ldots, x_n \in f(\mathbb{P}^1)$.
- There is a morphism $f : \mathbb{P}^1 \to X$ such that f^*T_X is ample.
- For every $x_1, \ldots, x_n \in X$ there is a morphism $f : \mathbb{P}^1 \to X$ such that f^*T_X is ample and $x_1, \ldots, x_n \in f(\mathbb{P}^1)$.

We say $X/\mathbb{R}$ is **geometrically rationally connected** if $X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$ is rationally connected over $\mathbb{C}$.

A **comb with n teeth** is a connected reduced curve C with irreducible components $D, C_1, \ldots, C_n$ with the following properties:

- D is smooth and every C_i is isomorphic to $\mathbb{P}^1$ (D is called the **handle** and the C_i the **teeth**);
- The only singularities of C are ordinary nodes;
- Every C_i intersects D at a single point x_i and $x_i \neq x_j$ for $i \neq j$. These are the only intersections between the irreducible components.

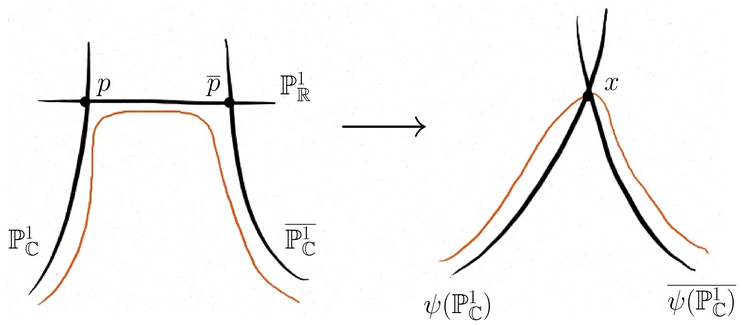
A **subcomb** of C is a subcurve which contains the handle.

Examples

- $\mathbb{P}^n$; two points determine a unique line
- unirational varieties; use the fact that $\mathbb{P}^n$ is rationally connected and pushforward
- smooth hypersurfaces $X \subset \mathbb{P}^n$ of degree $d \leq n$; follows from a theorem of Kollár, Miayoka and Mori that $-K_X$ ample implies rationally connected

Theorem [4] Let X be a smooth projective geometrically rationally connected variety over $\mathbb{R}$. Then X admits a non-constant morphism $\mathbb{P}^1_{\mathbb{R}} \to X$ if and only if $X(\mathbb{R}) \neq \varnothing$.

The idea of the proof is as follows. Let $x \in X(\mathbb{R})$. Then there exists a very free rational curve containing x, say $\psi : \mathbb{P}^1_{\mathbb{C}} \to X$ with $\psi(p) = x$. This means we may deform the curve to pass through any two prescribed points. Consider a copy of $\mathbb{P}^1_{\mathbb{R}}$ which is collapsed to our point x via some constant map $c : \mathbb{P}^1_{\mathbb{R}} \to x$. Then take a third mapping conjugate to the first. The map $\phi_0 = (\psi, c, \overline{\psi})$ on our chain of rational curves $\mathbb{P}^1 \cup \mathbb{P}^1 \cup \mathbb{P}^1$ defined over $\mathbb{R}$ (because complex conjugation exchanges the first and third components) collapses $\mathbb{P}^1_{\mathbb{R}}$ and gives $\psi(\mathbb{P}^1_{\mathbb{C}})$ and its conjugate $\overline{\psi(\mathbb{P}^1_{\mathbb{C}})}$ intersecting at x:



As a consequence of very freeness and using tools from deformation theory, we make the final claim that the image of ϕ_0 admits a smoothing ϕ over $\mathbb{R}$ with x contained in the image. This gives our map $\phi : \mathbb{P}^1_{\mathbb{R}} \to X$ with x in the image.

Finally, note that the existence of a rational point is necessary. If $\phi : \mathbb{P}^1_{\mathbb{R}} \to X$ exists then there is some $x \in X(\mathbb{R})$ in the image, say $\phi(y) = x$.

We use similar ideas to prove the following:

Proposition

Let X be a smooth projective rationally connected variety over $\mathbb{R}$ such that $X(\mathbb{R}) \neq \varnothing$. Let C be a smooth projective curve over $\mathbb{R}$. Then there exists a non-constant map $C \to X$.

Let $C \to x \in X$ be the constant map where $x \in X(\mathbb{R})$ is a real point. This gives a section $f : C \to X \times C$ of the natural projection $\pi_1 : X \times C \to C$. From here on we identify C with its image $f(C)$. Then, $\Gamma(C, \mathcal{N}_{C/X\times C}) \simeq \mathbb{R}^{\dim X}$ as an $\mathbb{R}$-vector space so the space of first order deformations is also $\mathbb{R}^{\dim X}$ by Theorem 1.2.4 [1]. This says that C may only be moved "up and down"and not otherwise deformed. In other words, no deformation fixes any point on C. We want to increase the size of the normal bundle, since first order deformations are identified with $H^0(C, \mathcal{N}_{C/X\times C})$. To do this, we construct a comb with handle C. Since each fiber $X_c \simeq X$ of $\pi_1 : X \times C \to C$ is rationally connected and smooth by hypothesis, there exists a very free rational curve through any point $x \in X$ by Corollary 4.28 [2]. Attaching enough very free $\mathbb{P}^1$'s through copies of x in different fibers guarantees our normal bundle is sufficiently large.

Then, there is a smoothing of a subcomb with at least one tooth fixing the $f(p_i)$ by Theorem II.7.9 [3]. Now we take such a smoothing and project the deformation of the handle down to X via $\pi_2 : X \times C \to X$. This gives the desired nontrivial map $\phi : C \to X$.

Proposition

Let C be a conic or a smooth projective curve such that $C(\mathbb{R}) = \varnothing$. Let M be a connected scheme over $\mathbb{R}$ and $\mathcal{X} \to M$ a flat projective morphism such that

- the generic fibers are smooth and geometrically rationally connected and all fibers have at worst isolated LCI terminal singularities;
- some smooth fiber $\mathcal{X}_m$ admits a morphism $\phi : C \to \mathcal{X}_m$.

Then every smooth fiber admits such a morphism.

Given a map to some smooth fiber $\phi : C \to \mathcal{X}_m$ defined over $\mathbb{R}$, we have a map over $\mathbb{C}$ such that $\phi = \overline{\phi}$. When we try to deform and get to a singular fiber the image of C may hit the singular locus. Attach enough conjugate $\mathbb{P}^1_{\mathbb{C}}$'s avoiding the singular locus and apply a smoothing result to deform past the singularity into the smooth fiber we care about.

References

[1] Robin Hartshorne, *Deformation Theory*, Graduate Texts in Mathematics, v. 257, Springer-Verlag, 2010.

[2] Oliver Debarre, *Higher Dimensional Algebraic Geometry*, Universitext, Springer-Verlag, 2001.

[3] János Kollár, *Rational Curves on Algebraic Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3.Folge, Band 32, Springer-Verlag, 1996.

[4] János Kollár, *Rationally connected varieties over local fields*, Annals of Mathematics, **150** (1999), 357-367.

[5] Tom Graber, Joe Harris, Jason Starr, *Families of rationally connected varieties*, Journal of the AMS, **16** 1 (2003), 57-67.