Stable Arakelov invariants of curves

Ariyan Javanpeykar (Johannes Gutenberg-Universität Mainz)

Advisors:
Jean-Benoît Bost (Paris-Sud 11) & Robin de Jong (Leiden University)

Main message

**Arakelov invariants are polynomial in the Belyi degree**

**Diophantine application**

Since our bounds are explicit, we can deduce **Szpiro's small points conjecture** (1985) for hyperelliptic curves. This is the first known case of Szpiro's conjecture.

**Geometric application: Covers of curves**

Let $K$ be a number field, let $Y$ be a smooth projective geometrically connected curve over $K$ and let $B$ be a finite set of closed points in $Y$. Then, if $p : X → Y$ is a finite morphism with branch locus contained in $B$, each Arakelov invariant of $X$ is polynomial in deg $p$; see the Corollary for a more precise statement when $Y=\mathbb{P}^1$.

**Stable Arakelov invariants of curves**

For a smooth projective geometrically connected curve $X$ (of positive genus) over a number field $K$, let $h(X)$ be its **stable Faltings height**. This is an **Arakelov invariant**.

The stable Faltings height plays an important role in Faltings' proof (1983) of the Mordell conjecture.

We mention two other Arakelov invariants of $X$.

The **stable discriminant** $D(X)$; this Arakelov invariant "measures" the bad reduction of $X$ over the ring of integers of $K$.

The **self-intersection of the dualizing sheaf** $e(X)$; this invariant plays a key role in the Bogomolov conjecture.

**The Belyi degree of a curve**

Let $K$ be a number field and let $\mathbb{P}^1$ be the projective line over $K$. For a curve $X$ over $K$ of positive genus, the **Belyi degree** of $X$, denoted by deg$_{Belyi}(X)$, is the minimal degree of a finite morphism $X → \mathbb{P}^1$ which ramifies over precisely three points.

**Finiteness property of the Belyi degree**

Let $\Omega/K$ be an algebraic closure of $K$. For any real number $c$, the set of $\Omega$-isomorphism classes of smooth projective connected curves $X/\Omega$ such that deg$_{Belyi}(X) < c$ is finite.

**Proof**. The topological fundamental group of $\mathbb{C}$-{0,1} is finitely generated. QED

Main theorem

For any smooth projective geometrically connected curve $X/K$,

$$h(X) < 10^9 \deg_{Belyi}(X)^7$$

$$D(X) < 10^9 \deg_{Belyi}(X)^7$$

$$e(X) < 10^9 \deg_{Belyi}(X)^7$$

**Corollary: Covers of the projective line**

Let $B$ be a finite set of closed points in $\mathbb{P}^1$. For any finite morphism $p : X → \mathbb{P}^1$ with branch locus contained in $B$,

$$h(X) < 10^9 (4bH_p)^{63b^4} (\deg p)^7$$

$$D(X) < 10^9 (4bH_p)^{63b^4} (\deg p)^7$$

$$e(X) < 10^9 (4bH_p)^{63b^4} (\deg p)^7$$

where $b = [K: \mathbb{Q}]#B$ and $H_p$ is the exponential height of $B$.

**Proving the main theorem**

Applying the arithmetic Hodge index theorem splits the proof of the main theorem into two parts: analytic and arithmetic.

**Analytic part**

We bound Arakelov-Green functions of unramified covers of the modular curve $Y(2)$ using a theorem of Merkl-Bruin.

**Arithmetic part**

We bound intersection numbers on a “wild” model of a Belyi morphism $X → \mathbb{P}^1$ over $\mathbb{Z}$ using Abhyankar’s lemma and Lenstra’s generalization of Dedekind’s discriminant bound.

**A conjecture of Edixhoven, de Jong and Schepers**

We were first led to investigate this problem by a conjecture of Edixhoven, de Jong and Schepers on the Faltings height. Their conjecture follows from the above corollary.

**Context: Étale cohomology of surfaces**

Let $S$ be a smooth projective geometrically connected surface over $\mathbb{Q}$. Let $\Omega$ be an algebraic closure of $\mathbb{Q}$. In view of our results, it seems reasonable to suspect that, following a strategy of Edixhoven, there is an algorithm that on input a prime $p$ computes, for $i=0,\ldots,4$, the cohomology groups $H^i(S_{\Omega,Q}, F_p)$ with their $Gal(\Omega/\mathbb{Q})$-action, in time polynomial in $p$. 
