Stable Arakelov invariants of curves

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Main message

Arakelov invariants are polynomial in the Belyi degree

Diophantine application

Since our bounds are explicit, we can deduce **Szpiro's small points conjecture** (1985) for hyperelliptic curves. This is the first known case of Szpiro's conjecture.

Geometric application: Covers of curves

Let K be a number field, let Y be a smooth projective geometrically connected curve over K and let B be a finite set of closed points in Y. Then, if $p : X \to Y$ is a finite morphism with branch locus contained in B, each Arakelov invariant of X is polynomial in deg p; see the Corollary for a more precise statement when $Y=\mathbf{P}^1$.

Stable Arakelov invariants of curves

For a smooth projective geometrically connected curve X (of positive genus) over a number field K, let $h(X)$ be its **stable Faltings height**. This is an *Arakelov invariant*.

The stable Faltings height plays an important role in Faltings' proof (1983) of the Mordell conjecture.

We mention two other Arakelov invariants of X.

The **stable discriminant** $D(X)$; this Arakelov invariant "measures" the bad reduction of X over the ring of integers of K.

The **self-intersection of the dualizing sheaf** $e(X)$; this invariant plays a key role in the Bogomolov conjecture.

The Belyi degree of a curve

Let K be a number field and let $\mathbf{P}^1$ be the projective line over K. For a curve X over K of positive genus, the **Belyi degree** of X, denoted by $\deg_{Belyi}(X)$, is the minimal degree of a finite morphism $X \to \mathbf{P}^1$ which ramifies over precisely three points.

Finiteness property of the Belyi degree

Let Ω/K be an algebraic closure of K. For any real number c, the set of Ω-isomorphism classes of smooth projective connected curves X/Ω such that $\deg_{Belyi}(X) < c$ is finite.

Proof. The topological fundamental group of $\mathbf{C}$-{0,1} is finitely generated. QED

Main theorem

For any smooth projective geometrically connected curve X/K,

$$h(X) < 10^9 \deg_{Belyi}(X)^7$$

$$D(X) < 10^9 \deg_{Belyi}(X)^7$$

$$e(X) < 10^9 \deg_{Belyi}(X)^7$$

Corollary: Covers of the projective line

Let B be a finite set of closed points in $\mathbf{P}^1$. For any finite morphism $p : X \to \mathbf{P}^1$ with branch locus contained in B,

$$h(X) < 10^9 (4bH_B)^{63\, b!!} (\deg p)^7$$

$$D(X) < 10^9 (4bH_B)^{63\, b!!} (\deg p)^7$$

$$e(X) < 10^9 (4bH_B)^{63\, b!!} (\deg p)^7,$$

where $b = [K:\mathbf{Q}]\#B$ and H_B is the exponential height of B.

Proving the main theorem

Applying the arithmetic Hodge index theorem splits the proof of the main theorem into two parts: analytic and arithmetic.

Analytic part

We bound Arakelov-Green functions of unramified covers of the modular curve $Y(2)$ using a theorem of Merkl-Bruin.

Arithmetic part

We bound intersection numbers on a "wild" model of a Belyi morphism $X \to \mathbf{P}^1$ over $\mathbf{Z}$ using Abhyankar's lemma and Lenstra's generalization of Dedekind's discriminant bound.

A conjecture of Edixhoven, de Jong and Schepers

We were first led to investigate this problem by a conjecture of Edixhoven, de Jong and Schepers on the Faltings height. Their conjecture follows from the above corollary.

Context: Étale cohomology of surfaces

Let S be a smooth projective geometrically connected surface over $\mathbf{Q}$. Let Ω be an algebraic closure of $\mathbf{Q}$. In view of our results, it seems reasonable to suspect that, following a strategy of Edixhoven, there is an algorithm that on input a prime p computes, for i=0,..,4, the cohomology groups $H^i(S_{\Omega,ét}, \mathbf{F}_p)$ with their Gal($\Omega/\mathbf{Q}$)-action, in time polynomial in p.