Definable Operations On Weakly Recognizable Sets of Trees

Jacques Duparc[1], Alessandro Facchini[2], and Filip Murlak[3]

1 University of Lausanne, Switzerland
jacques.duparc@unil.ch
2 University of Warsaw, Poland
facchini@mimuw.edu.pl
3 University of Warsaw, Poland
fmurlak@mimuw.edu.pl

Abstract

Alternating automata on infinite trees induce operations on languages which do not preserve natural equivalence relations, like having the same Mostowski–Rabin index, the same Borel rank, or being continuously reducible to each other (Wadge equivalence). In order to prevent this, alternation needs to be restricted to the choice of direction in the tree. For weak alternating automata with restricted alternation a small set of computable operations generates all definable operations, which implies that the Wadge degree of a given automaton is computable. The weak index and the Borel rank coincide, and are computable. An equivalent automaton of minimal index can be computed in polynomial time (if the productive states of the automaton are given).

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1 Introduction

The structure of a regular language of infinite trees can be analyzed in terms of recognizing automata, defining formulas, or topological properties. Each approach defines a hierarchy of classes of similar languages: the Mostowski–Rabin index hierarchy, the μ-calculus alternation hierarchy, the Borel hierarchy, the Wadge hierarchy. Sometimes complementary, sometimes closely related, together they approximate the missing canonical representation of regular languages. Understanding them has been a goal pursued for decades, bringing spectacular successes like the Wagner hierarchy for regular languages of infinite words [24], providing a full characterization of the topological and combinatorial structure of a language in terms of certain patterns in the recognizing deterministic automaton. Various versions of this pattern method were successfully applied to deterministic automata on trees, resulting in a full classification in terms of Wadge equivalence [16, 18], non-deterministic index [19], and weak alternating index [17].

Owing to the elegant correspondence between certain set-theoretical and ordinal operations [4], the whole Wadge hierarchy of Borel sets of finite rank can be generated with several simple operations, starting from the empty set. The pattern method builds on this result. In order to obtain lower bounds for the Wadge hierarchy of the considered class of automata, it is often enough to check that some operations are definable within the class [5, 6].

In obtaining upper bounds and computability results, the pattern method relies on certain compositionality of deterministic automata with respect to the equivalence relations of having

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the same Mostowski–Rabin index, the same Borel rank, or being continuously reducible to each other (Wadge equivalence). In a deterministic automaton each sub-automaton can be replaced with any automaton recognizing an equivalent language without influencing the equivalence class of the whole language. More generally, each automaton can be seen as a result of an operation performed on sub-automata by means of some connecting automaton. If the connecting automaton is deterministic, the operation induces an operation on the equivalence classes of the corresponding languages (see [16, 18], and also [19]). Sometimes, these operations can be expressed in terms of computable ordinal operations on Wadge degrees, and the degree of the recognized language can be obtained by bottom up evaluation starting from the simple sub-automata [16].

For alternating automata this approach fails in general, because the set-theoretical operation of union, easily simulated within an alternating automaton, is not an operation on the equivalence classes. Indeed, the union of arbitrarily complicated languages can be the whole space. Does it mean that the pattern method is confined to deterministic automata? Recently it has been shown that the method can be extended beyond deterministic automata, but the class of considered languages was very small [7]. In this paper we introduce a large syntactic class of the automata inducing operations compatible with the Wadge equivalence—we call them *game automata*—and show that it is the largest such class satisfying natural closure conditions (Sect. 4). We then focus on weak automata, and identify a small set of operations on Wadge equivalence classes which generate all other definable operations (Sect. 5). Based on this we show how to compute the Wadge degree and the Borel rank of weak game automata (Sect. 6). Finally, we prove that the Borel rank and the weak index coincide for weak game automata, which leads to an algorithm computing the weak index, and constructing the equivalent automaton with minimal index (Sect. 7).

Due to space limitations many proofs are moved to the full version of the paper [8].

2 Alternating Tree Automata

Let T_Σ denote the set of *(full infinite binary) trees* over an alphabet Σ, i.e., functions $t : \{0,1\}^* \to \Sigma$. Given $v \in \mathsf{dom}(t)$, by $t.v$ we denote the subtree of t rooted in v.

A *alternating tree automaton* $\langle \Sigma, Q, q_I, \delta, \mathsf{rank} \rangle$ consisting of a finite alphabet Σ, a finite set of states Q, an initial state $q_I \in Q$, a transition function $\delta : Q \times \Sigma \to \mathcal{B}^+(\{0,1\} \times Q)$, where $\mathcal{B}^+(\{0,1\} \times Q)$ denotes the set of positive boolean formulae over $\{0,1\} \times Q$, and a rank function $\mathsf{rank} : Q \to \mathbb{N}$. As usual, A accepts $t \in T_\Sigma$ iff the player $\Diamond$ has a winning strategy in the induced max-parity game (see [8] for details). To underline this connection, we write transitions with $\square$ and $\diamond$ instead of $\wedge$ and $\vee$, e.g., $\delta(p,\sigma) = ((0,p) \square (0,q)) \diamond (1,q)$, or $p \xrightarrow{\sigma} ((0,p) \square (0,q)) \diamond (1,q)$. The class of all alternating automata is denoted by ATA.

An alternating tree automaton A is

- *weak* (WATA), if for all $q, q' \in Q$, if q is reachable from q' and q' is reachable from q, then $\mathsf{rank}(q) = \mathsf{rank}(q')$;
- *linear* (LATA), if for all $q, q' \in Q$, if q is reachable from q' and q' is reachable from q, then $q = q'$, and for each $q \in Q$ either all $\delta(q,\sigma)$ use only $\square$ or all use only $\diamond$;
- *deterministic* (DTA), if for all $q \in Q$, $\sigma \in \Sigma$, $\delta(q,\sigma) \in \{(0,p) \square (1,q) \mid p, q \in Q\}$.

A state q is *reachable* from p if there exists a *path* in A from p to q, i.e., a sequence of states and alphabet symbols $p_0\sigma_0p_1\sigma_1 \dots p_{k-1}\sigma_k p_k$ such that $p_0 = p$, $p_k = q$, and p_{i+1} occurs in $\delta(p_i, \sigma_i)$ for all $i < k$. Throughout the paper we assume that all states are reachable from the initial state. By convention, $\top$ is a singled out all-accepting state, and $\bot$ is an all-rejecting state. We assume that all other states are *non-trivial*, i.e., accept some tree and reject some tree. For every state q which is not the initial state q_I of the automaton A, by A_q we denote the automaton corresponding exactly to A except the fact that the initial state now is q and not q_I. We say that a state q of A is *productive* if $L(A_q) \neq \varnothing$.

The *(Mostowski–Rabin) index of an automaton* is given by $(i, j) \in \{0, 1\} \times \omega$, where i is the minimal and j is the maximal value of rank (scaling down the priorities we can always assume that the smallest rank is 0 or 1). Classes of languages recognizable with automata of index (i, j) form the so-called *index hierarchy*. By a result of Bradfield [3], we know that the index hierarchy for alternating tree automata, is strict. It is well-known that the class of weakly recognizable languages forms a strict hierarchy with respect to the index of the recognizing weak automata (cf. [1]). In the latter case we speak of the *weak index hierarchy*.

3 Borel classes and Wadge reductions

Consider the space T_Σ equipped with the standard Cantor (prefix) topology, that is the topology where a basic open set is the set all trees that extend a certain finite tree. Recall that the class of Borel sets of a topological space X is the closure of the class of open sets of X by countable unions and complementation. For a topological space X, the initial finite levels of the Borel hierarchy are defined as follows:

- $\mathbf{\Sigma}^0_1(X)$ is the class of open subsets of X,
- $\mathbf{\Pi}^0_n(X)$ contains complements of sets from $\mathbf{\Sigma}^0_n(X)$,
- $\mathbf{\Sigma}^0_{n+1}(X)$ contains countable unions of sets from $\mathbf{\Pi}^0_n(X)$.

By convention $\mathbf{\Sigma}^0_0(X) = \{\varnothing\}$ and $\mathbf{\Pi}^0_0(X) = \{X\}$.

The classes defined above are closed under inverse images of continuous functions. Let $\mathcal{C}$ be one of those classes. A set U is called $\mathcal{C}$-hard, if each set in $\mathcal{C}$ is an inverse image of U under some continuous function. If additionally $U \in \mathcal{C}$, U is said to be $\mathcal{C}$-complete. It is well known that every weakly recognizable tree language is a member of a Borel class of finite rank ([6, 14]). The rank of a language is the rank of the minimal Borel class the language belongs to. It can be seen as a coarse measure of complexity of languages.

A much finer measure of the topological complexity is the *Wadge degree*. If $T, U \subseteq T_\Sigma$, we say that T is *continuously (or Wadge) reducible* to U, $T \leqslant_W U$ in symbols, if there exists a continuous function f such that $T = f^{-1}(U)$. For a Borel class $\mathcal{C}$, T is $\mathcal{C}$-hard if $U \leqslant_W T$ for every $U \in \mathcal{C}$. We write $T \equiv_W U$ whenever $T \leqslant_W U \leqslant_W T$, and $T <_W U$, if $T \leqslant_W U$ but not $U \leqslant_W T$. The *Wadge hierarchy* is the partial order induced by $<_W$ on the $\equiv_W$-equivalence classes of Borel sets.

An alternative characterization of continuous reducibility can be given in terms of games. Let T and U be two arbitrary sets of trees. The *Wadge game* $\mathcal{W}(T, U)$ is played by two players, player I and player II. Each player builds a tree, say t_{I} and t_{II}, level by level. In every round, player I plays first, and both players add one level to their trees. Player II is allowed to skip her turn, but not forever. Player II wins the game if $t_{\text{I}} \in T \Leftrightarrow t_{\text{II}} \in U$.

▸ **Lemma 1** ([23]). *Let $T, U \subseteq T_\Sigma$. Then $T \leqslant_W U$ iff Player II has a winning strategy in the game $\mathcal{W}(T, U)$.*

An ordinal number is the order type of a well-ordered set. The least infinite ordinal is denoted by ω and corresponds to the order-type of the set of all natural numbers. We say that an ordinal α is countable if there is a bijection between α and ω. The first uncountable ordinal is denoted by ω_1. A subset B of an ordinal α is said to be *cofinal* if for every $a \in \alpha$ there exists some $b \in B$ such that $a \in b$. The *cofinality* of an ordinal α is thence the smallest ordinal β that is the order type of a cofinal subset of α.

Recall that a language L is called *self dual* if it is equivalent to its complement, otherwise it is called *non self dual.* From Borel determinacy [13], if $T, U \subseteq T_\Sigma$ are Borel, then $\mathcal{W}(T, U)$ is determined. As a consequence, a variant of Martin-Monk's result (cf. [11]) shows that $<_W$ is well-founded. Thus, we can associate to every Borel language an ordinal, called the *Wadge degree*, i.e. for sets of finite Borel rank, their Wadge degree is inductively defined by:

- $d_W(\varnothing) = d_W(\varnothing^\complement) = 1$,
- $d_W(L) = \sup\{d_W(K) + 1 \colon K \text{ non self dual}, K <_W L\}$ for $L >_W \varnothing$, non self-dual,
- $d_W(L) = \sup\{d_W(K) \colon K \text{ non self dual}, K <_W L\}$ for L self-dual.

For instance, open, non-closed sets have degree 2, just like closed, non-open sets. All clopens have degree 1. Let $\exp(\alpha) = \omega_1^\alpha$, and let ${}^{\omega_1}\epsilon_0 = \sup_{n\in\omega} \exp^n(\omega_1)$, the least fixpoint of the ordinal exponentiation of base ω_1. This is known to be the height of the Wadge hierarchy of all tree languages (recognizable or not) of finite Borel rank. More precisely, if L is $\mathbf{\Sigma}^0_n$-complete for $n > 1$, then $d_W(L) = \exp^{n-1}(1)$ for $n > 1$ (cf. [4]).

For each degree there are exactly three equivalence classes with the same degree, represented by U, $U^\complement$ and $U^\pm = \{t \mid t(\epsilon) = a, t.0 \in U\} \cup \{t \mid t(\epsilon) \neq a, t.0 \notin U\}$ for some non self-dual set U and $a \in \Sigma$. It easy to check that $U, U^\complement <_W U^\pm$ and $U^\pm$ is self-dual.

For each non self-dual set one can determine its sign, $+$ or $-$, which specifies precisely the $\equiv_W$-class [4]. For sets $U \subseteq T_\Sigma$ with $d_W(U)$ of countable cofinality, the sign is $+$ if U is Wadge equivalent to the set of trees over $\Sigma \cup \{c\}$, $c \notin \Sigma$, which have no c on the leftmost branch, or the first c is in the node 0^i and $t.0^i \in U$. The sign is $-$ if U is equivalent to the complement of this set. For instance, $\varnothing$ and open, non-closed sets have sign -, while the whole space and closed, non-open sets have sign $+$. For sets of cofinality ω_1, the definition is more complicated, but $\mathbf{\Sigma}^0_n$-complete sets have sign $-$, and $\mathbf{\Pi}^0_n$-complete sets have sign $+$. All self-dual sets by definition have sign $\pm$. Thus an ordinal $\alpha < {}^{\omega_1}\epsilon_0$ and a sign $\epsilon \in \{+, -, \pm\}$, determine a $\equiv_W$-class, denoted $[\alpha]^\epsilon$.

4 Game automata

For A, B (over the same alphabet) and an occurrence of a state q in a transition $\delta(p, \sigma)$ of A, the substitution A_B is obtained by replacing the occurrence of q in $\delta(p, \sigma)$ with the initial state of B. The mapping $B \mapsto A_B$ induces an operation on recognized languages, but it need not preserve coarser equivalence relations, like Wadge equivalence.

As pointed out in the introduction, the operation of union is not compatible with such equivalence relations. The same is true of intersection.

▸ **Example 2.** Take $\Sigma = \{0, 1, 2\}$ and consider $(\Sigma^*(1 + 2))^\omega$ and $(\Sigma^*2)^\omega$. Clearly, $(\Sigma^*(1 + 2))^\omega \leqslant_W (\Sigma^*2)^\omega$ as witnessed by the letter-to-letter morphism $0 \mapsto 0$ and $1, 2 \mapsto 2$. The converse reduction is given by the inclusion. Taking union with Σ^*0^ω, we obtain $(\Sigma^*(1 + 2))^\omega \cup \Sigma^*0^\omega = \Sigma^\omega$, and $(\Sigma^*2)^\omega \cup \Sigma^*0^\omega \not\equiv_W \Sigma^\omega$. The language $(\Sigma^*2)^\omega \cup \Sigma^*0^\omega$ is at the level $\mathbf{\Delta}^0_3$ of the Borel hierarchy, a deterministic automaton requires three ranks to recognize it, and an alternating automaton needs two. This makes it much more complex than the whole space Σ^ω, which can be recognized by a deterministic automaton with a single state, whose rank is 0. Similarly, intersecting with $\Sigma^*(0 + 1)^\omega$ we obtain $\Sigma^*(0^*1)^\omega$, and the empty set, which have very different complexity.

In order to ensure that substitution is well-behaved, we need to prevent the automata from simulating union and intersection. We call a transition $\delta(q, a)$ *ambiguous* if it contains two occurrences of some direction $d \in \{0, 1\}$.

▸ **Fact 3.** Let $\mathcal{C} \subseteq \mathsf{ATA}$ be a class of automata over a fixed alphabet with at least two letters, closed under substitution and containing the one-state all-rejecting and all-accepting automata. Substitution preserves the Wadge equivalence in $\mathcal{C}$ iff no automaton of $\mathcal{C}$ has an ambiguous transition.

Proof. Assume for simplicity that the alphabet contains the symbols $0, 1, 2$. Starting from the all-accepting and all-rejecting automata over the alphabet $\{0, 1, 2\}$ we can obtain automata $A, A^{\complement}B, B^{\complement}$ recognizing languages L_{0^ω}, $(L_{0^\omega})^{\complement}$, $L_{(0+1)^\omega}$, $(L_{(0+1)^\omega})^{\complement}$ respectively, where L_α stands for the set of trees whose leftmost branch is a word from the language defined by the expression α. Observe that $L(A) \equiv_W L(B)$, but $L(A) \cup L(B^{\complement}) \not\equiv_w L(B) \cup L(B^{\complement})$ and $L(A) \cap L(A^{\complement}) \not\equiv_w L(B) \cap L(A^{\complement})$.

Let $C \in \mathcal{C}$ and let $q_0\sigma_0q_1\sigma_1 \dots q_k$ be path from the initial state q_0 to a state q_k such that for some σ_k, $\delta(q_k, \sigma_k)$ is an ambiguous transition. By substituting the all-accepting and all-rejecting automata, we can assume that $\delta(q_i, \sigma_i) = (d_i, q_{i+1})$ for $i < k$ and $\delta(q_k, \sigma_k) = (d_k, p_0) \diamond (d_k, p_1)$ or $\delta(q_k, \sigma_k) = (d_k, p_0) \Box (d_k, p_1)$ for some states p_0, p_1. Assume that $\delta(q_k, \sigma_k) = (d_k, p_0) \diamond (d_k, p_1)$, and let C' be the result of replacing the occurrence of p_0 with the initial state of B, and the occurrence of p_1 with the initial state of $B^{\complement}$. For C'_A, obtained by replacing the initial state of B with the initial state of A, we have $L(C'_A) \equiv_W L(A) \cup L(B^{\complement})$, and $L(C') \equiv_W L(B) \cup L(B^{\complement})$, which concludes the proof. For $\delta(q_k, \sigma_k) = (d_k, p_0) \Box (d_k, p_1)$, use $A^{\complement}$ instead of $B^{\complement}$. ◂

Observe that each non-ambiguous transition has one of the four forms: $(0, p)$, $(1, p)$, $(0, p) \diamond (1, q)$, or $(0, p) \Box (1, q)$.

▸ **Definition 4.** A *game automaton* (GA) is an alternating automaton without ambiguous transitions. For notational simplicity, we assume that

$$\delta \colon Q \times \Sigma \to \{p \diamond q \mid p, q \in Q \backslash \{\top\}\} \cup \{p \Box q \mid p, q \in Q \backslash \{\bot\}\} ,$$

where $p \diamond q$ and $p \Box q$ is interpreted as $(0, p) \diamond (1, q)$ and $(0, p) \Box (1, q)$, respectively.

A *weak game automaton* (WGA), is a game automaton which is also weak, and a *linear game automaton* (LGA) [7], is a game automaton which is linear.

Fact 3 implies that GA it the largest nontrivial subclass of ATA closed under substitution for which substitution preserves Wadge equivalence, and similarly for $\mathsf{WGA} \subseteq \mathsf{WATA}$. In fact, a more general property holds for GA.

▸ **Fact 5.** For every GA A, B, B', every state q of A, and A_B, $A_{B'}$ obtained by replacing an occurrence of q with the initial state of B and B' respectively, it holds that

1. $L(B) \leqslant_W L(B')$ implies $L(A_B) \leqslant_W L(A_{B'})$,
2. $L(A_q) \leqslant_W L(A)$.

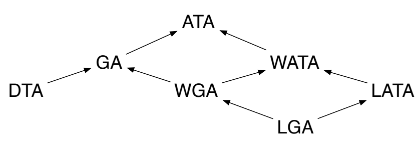

Figure 1

Relations between the classes are shown in Fig. 1 with arrows standing for class inclusion. The classes GA, WGA, and LGA are closed under complementation: the usual complementation procedure of increasing the ranks by one and swapping existential and universal transitions works. However they are neither closed under union nor intersection. For instance, let $L_\sigma = \{t \in T_{\{a,b\}} \colon t(0) = t(1) = \sigma\}$. Obviously, L_a and L_b are LGA-recognizable, but $L_a \cup L_b$ is not even GA-recognizable. Note that the last example also shows that all the inclusions in the diagram above are strict.

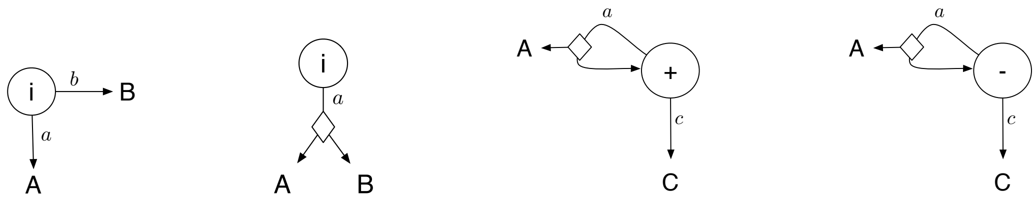

Figure 2 Automata constructions for $\sqcup$, $\diamond$, loop^+, $\exists$.

5 Operations induced by automata

LGA, investigated in [7], can be classified in terms of several simple set theoretic operations (we assume that the alphabet contains letters a, b, c):

$$\begin{aligned}
L \sqcup M &= \{t \mid t(\varepsilon) = a, t.0 \in L\} \cup \{t \mid t(\varepsilon) \neq a, t.0 \in M\} \ ,\\
L \mathbin{\square} M &= \{t \mid t.0 \in L \wedge t.1 \in M\} \ ,\\
L \diamond M &= \{t \mid t.0 \in L \vee t.1 \in M\} \ ,\\
\mathsf{loop}^-(L, M) &= \bigcup_{n\in\mathbb{N}} \{t \mid \text{first } c \text{ is in } 0^n, t.0^{n+1} \in M, \text{ and } t.0^\ell 1 \in L \text{ for all } \ell < n\} \ ,\\
\mathsf{loop}^+(L, M) &= \bigcup_{n\in\mathbb{N}} \{t \mid \text{first } c \text{ is in } 0^n, \text{ and } t.0^{n+1} \in M \text{ or } t.0^\ell 1 \in L \text{ for some } \ell < n\} \cup\\
&\quad \cup \{t \mid t.(0^n) \neq c \text{ for all } n\} \, ,\\
\forall(L, M) &= \mathsf{loop}^-(L, M) \cup \{t \mid t.(0^n) \neq c \text{ for all } n, \text{ and } t.0^\ell 1 \in L \text{ for all } \ell\} \, ,\\
\exists(L, M) &= \mathsf{loop}^+(L, M) \cup \{t \mid t.(0^n) \neq c \text{ for all } n, \text{ and } t.0^\ell 1 \in L \text{ for some } \ell\} \, ,
\end{aligned}$$

where "first c is in 0^n" means that $t(0^n) = c$ and $t(0^k) \neq c$ for all $k < n$. Observe that $(L \mathbin{\square} M)^\complement = L^\complement \diamond M^\complement$, $(\mathsf{loop}^+(L, M))^\complement = \mathsf{loop}^-(L^\complement, M^\complement)$, and $(\forall(L, M))^\complement = \exists(L^\complement, M^\complement)$.

These operations are definable by LGA: automata realizations for $\sqcup$, $\diamond$, loop^+, $\exists$ are shown in Fig. 2, and for $\square$, loop^-, $\forall$ they are obtained by replacing $\diamond$ with $\square$ and swapping the rank parities. Like all operations induced by GA, they are compatible with Wadge equivalence.

▸ **Fact 6.** Let op be one of the operations $\sqcup$, $\diamond$, loop^+, $\exists$, or their duals. Whenever $L \equiv_W L'$ and $M \equiv_W M'$, it holds that $\mathsf{op}(L, M) \equiv_W \mathsf{op}(L', M')$.

Up to Wadge equivalence, these operations are everything LGA are able to express.

▸ **Fact 7** ([7])**.** Up to Wadge equivalence, the closure of $\{\top, \bot\}$ under $\sqcup$, $\diamond$, loop^+, $\exists$, and their duals (or equivalently, complementation) gives exactly the family of sets recognized by LGAs. Moreover, for each LGA one can compute an equivalent canonical term over these operations and $\bot$, $\top$.

Since the operations preserve Wadge equivalence, they can be defined in terms of ordinal arithmetics and signs [4, 7]. For some operations the definitions are very simple, for instance

$$[\gamma_1]^{\epsilon_1} \sqcup [\gamma_2]^{\epsilon_2} = [\max(\gamma_1, \gamma_2)]^\epsilon, \text{ where } \epsilon = \begin{cases} \epsilon_1 & \text{if } \gamma_1 > \gamma_2 \\ \pm & \text{if } \gamma_1 = \gamma_2 \text{ and } \epsilon_1 \neq \epsilon_2, \\ \epsilon_2 & \text{otherwise} \end{cases}$$

$$\mathsf{loop}^+([\gamma]^\epsilon, [1]^-) = \Big[\sup_k d_W(([\gamma]^\epsilon)^{\langle k\rangle})\Big]^+ \, , \text{ where } U^{\langle k\rangle} = \underbrace{U \diamond U \diamond \cdots \diamond U}_{k}$$

$$\exists([\gamma]^\epsilon, [1]^-) = [\exp^{i+1} 1]^-, \text{ for } [\exp^i 1]^+ \leqslant_W [\gamma]^\epsilon \leqslant_W [\exp^{i+1} 1]^- \, .$$

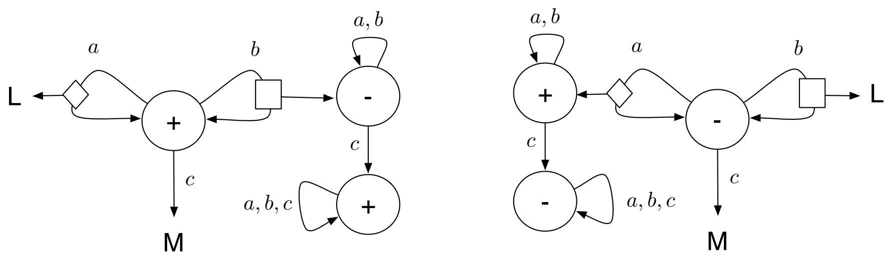

Figure 3 Operations definable with WGA.

Observe that the second equation, and its dual, imply that for all k

$$\mathsf{loop}^+(L, M) \geqslant_W L^{\langle k \rangle}\,, \qquad \mathsf{loop}^-(L, M) \geqslant_W L^{[k]}\,,$$

where $U^{[k]} = \underbrace{U \,\square\, U \,\square \cdots \square\, U}_{k}$. For $\diamond$ the ordinal definition has only been given for $\equiv_W$-classes inhabited by LGA-recognizable languages, $[\Phi] = \{[\alpha]^\epsilon \mid \alpha \in \Phi, \epsilon \in \{+,-,\pm\}\}$ with Φ denoting the set of ordinals of the form $\sum_{n=N}^{0} \beta_n + \alpha$ where $\alpha < \omega$ and each β_n is of the form $\exp^n(\omega)\eta + \sum_{p=P}^{1} \exp^n(p)k_p$ for some $\eta < \omega^\omega$ and $k_p < \omega$. Closure of $[\Phi]$ under $\sqcup$, $\diamond$, loop^+, $\exists$ (and their duals) was the technical core of the proof of Fact 7.

In this work we want to move to sets recognizable by WGA. Surprisingly, only two really new operations are introduced, $\mathsf{loop\text{-}reset}^+(L, M)$ and $\mathsf{loop\text{-}reset}^-(L, M)$. The automata constructions for them are shown in Fig. 3.

By a Wadge game argument we get a simple characterization in terms of ordinal arithmetics, showing that WGA-definable operations can multiply some Wadge degrees by ω_1.

▸ **Theorem 8.** *For every Wadge equivalence class $[\gamma]^\epsilon$ of a Borel language and $\mu \in \{+,-\}$*

$$\mathsf{loop\text{-}reset}^\mu([\gamma]^\epsilon, [1]^{\overline{\mu}}) = \begin{cases} [3]^{\overline{\mu}} & \text{if } [\gamma]^\epsilon \equiv_W [1]^\mu, \\ \left[d_W(\mathsf{loop}^+([\gamma]^\epsilon, [1]^-))\omega_1\right]^\mu & \text{otherwise}, \end{cases}$$

where $\overline{\mu} = +$ if $\mu = -$, $\overline{\mu} = -$ otherwise.

This operation is the source of difference between LGA and WGA, and allows WGA to inhabit many more Wadge equivalence classes than LGA. Thus, in our algorithm for WGA we use effective closure for a larger set of ordinals. Let Ω be the set of ordinals of the form $\Sigma_{i=K}^{0} \exp(\alpha_i)\eta_i$ where $\alpha_K, \alpha_{K-1}, \ldots, \alpha_0$ is a strictly decreasing sequence of ordinals from Φ, and $\eta_i < \omega$ for $\mathrm{cof}\,\alpha_i = \omega_1$ or $\mathrm{cof}\,\alpha_i < \omega$, and $\eta_i < \omega^\omega$ for $\mathrm{cof}\,\alpha_i = \omega$.

▸ **Lemma 9.** *$[\Omega]$ is closed under the operations $\sqcup$, loop^+, $\mathsf{loop\text{-}reset}^+$, $\exists$ (and their duals) and the result of the operation can be computed effectively.*

The proof is by induction, with the base cases covered by the closure property for $[\Phi]$.

6 Computing the Wadge degrees of WGA

For game automata, a run (computation tree) over an input tree t is a labeling of the input tree with states and modes ($\square$ or $\diamond$), induced by the transition function of the automaton. A transition taken from a node v determines the mode of v and the states in its children as follows: the root is labelled with the initial state, and if a node with label σ gets state q and

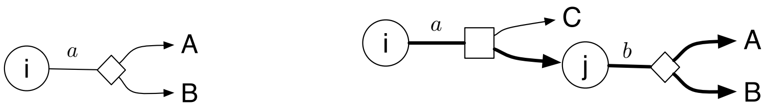

Figure 4 A simulation (the rank j must not be greater then i).

$q \xrightarrow{\sigma} q' \circ q''$ then v gets the mode $\circ$, and the left and right children get the states q' and q'' respectively. A run ρ is *resolved up to* a subtree ρ' if for all $v, v0, v1 \in \mathsf{dom}\,\rho$ such that exactly one node vd belongs to $\mathsf{dom}\,\rho'$, and for the remaining node vd' the sub-run $\rho.vd'$ is accepting if v's mode is $\square$ and rejecting if it is $\diamond$.

▸ **Definition 10.** A *simulation* of a run ρ in a run σ is a partial function $\eta : \mathsf{dom}\,\rho \to \mathsf{dom}\,\sigma$ such that

- $\mathsf{dom}\,\eta$ is a prefix closed subset of $\mathsf{dom}\,\rho$ (possibly with leaves and infinite branches);
- σ is resolved up to the subtree induced by the image of η;
- for each $v0, v1 \in \mathsf{dom}\,\eta$, $\eta(v0), \eta(v1)$ are descendants of $\eta(v)$, their closest common ancestor has the same mode as v , and the highest rank on the path from $\eta(v)$ to $\eta(vd)$ is equal to the rank of state in vd for $d = 0, 1$;
- for each leaf $v \in \mathsf{dom}\,\eta$, $\rho.v$ is accepting iff $\sigma.\eta(v)$ is accepting.

▸ **Lemma 11.** *If there is a game simulation of ρ in σ, then ρ is accepting iff σ is accepting.*

Proof. Each strategy in the parity game on ρ can be carried over to σ, and *vice versa.* ◂

▸ **Definition 12.** A *simulation* of an automaton A in an automaton B consists of a partition of Q^A into sets Q_1, Q_2, Q_3 and function $\eta\colon Q_1 \cup Q_2 \to Q^B$ such that

- $q_I^A \in Q_1$ and each transition of A originating in Q_1 leads to $Q_1 \cup Q_2$;
- whenever $q \xrightarrow{\sigma}_A q_0 \circ q_1$ for some $q \in Q_1$ and $\circ \in \{\diamond, \square\}$, there exist a path π from $\eta(q)$ to some p and paths π_i from some p_i to $\eta(p_i)$ for $i = 0, 1$ such that $p \xrightarrow{\tau}_B p_0 \circ p_1$ or $p \xrightarrow{\tau}_B p_1 \circ p_0$ and the highest rank on $\pi\tau\pi_i$ is equal to $\mathsf{rank}\, q_i$;
- for all $q \in Q_2$, $L(A_q) \leqslant_W L(B_{\eta(q)})$.

An example of a simulation is given in Fig. 4. A simulation of A in B immediately provides a continuous reduction from the set of accepting runs of A to the set of accepting runs of B. The next lemma follows by noticing that for GAs the set of accepting runs is Wadge equivalent to the recognized language.

▸ **Lemma 13.** *If there exists a simulation of A in B, then $L(A) \leqslant_W L(B)$.*

Strongly connected components (SCCs) of automata are defined as for graphs in terms of reachability. An SCC is *trivial* if it does not contain any loop. A transition $q \xrightarrow{\sigma} q' \circ q''$ is called *branching* if q, q', q'' belong to the same SCC.

▸ **Lemma 14.** *For each* WGA *one can effectively compute a Wadge equivalent* WGA *over $\{a, b, c\}$ without non-trivial loops.*

Proof. First we construct an automaton over a larger alphabet. We collapse each strongly connected component into one state, proceeding by induction on the DAG of SCCs. Let X be the root SCC, i.e., the SCC containing the initial state q_I. By induction hypothesis, we can assume that all other SCCs consist of a single state.

If there is a branching $\square$-transition in X, set $q_I \xrightarrow{a} q_I \square q_I$. Otherwise, set $q_I \xrightarrow{a_p} q_I \square p$ for all $p \notin X$ such that $q \xrightarrow{\sigma} q' \square p$ or $q \xrightarrow{\sigma} p \square q'$ for some $q, q' \in X$. Define the transitions via b and b_p analogously, replacing $\square$ with $\diamond$. Finally, let $q_I \xrightarrow{c_{p\circ p'}} p \circ p'$ where $p \circ p'$ ranges over

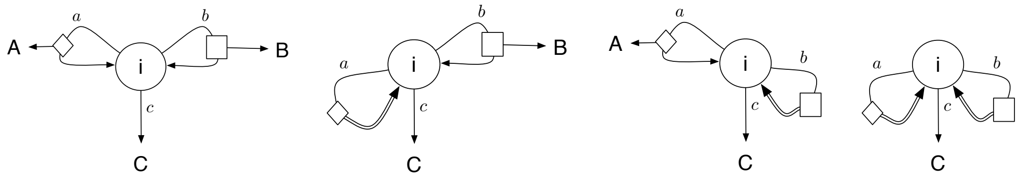

Figure 5 Strongly connected components of WGA over $\{a, b, c\}$ without non-trivial loops.

- $p \circ p'$ such that $p, p' \notin X$ and $q \xrightarrow{\sigma} p \circ p'$ for some $q \in X$, $\circ \in \{\diamond, \square\}$;
- $p \square \top$ such that $p \notin X$ and $q \xrightarrow{\sigma} q' \square p$ or $q \xrightarrow{\sigma} p \square q'$ for some $q, q' \in X$; and
- $p \diamond \bot$ such that $p \notin X$ and $q \xrightarrow{\sigma} q' \diamond p$ or $q \xrightarrow{\sigma} p \diamond q'$ for some $q, q' \in X$.

For each state q of the new automaton, extend $\delta(q, \sigma)$ to all symbols in the alphabet by using one of already defined transitions.

The original automaton can be simulated in the modified one by taking $Q_1 = X$, $Q_2 = \{p \mid q \xrightarrow{\sigma} p \circ r \text{ or } q \xrightarrow{\sigma} r \circ p \text{ for some } q \in X, r \in Q\}$, and $\eta(q) = q_I$ for $q \in Q_1$, and $\eta(p) = p$ for $p \in Q_2$. For the converse simulation, only change Q_1 to $\{q_I\}$, and for Q_2 and η keep the definitions above.

To reduce the alphabet to $\{a, b, c\}$, modify the construction as follows. In the case where there is no branching $\square$-transition in X, add a single transition $q_I \xrightarrow{a} q_I \square q_a$, where q_a is the initial state of the automaton recognizing $L(A_{p_1}) \sqcup L(A_{p_2}) \sqcup \cdots \sqcup L(A_{p_k})$, where $\{p_1, p_2, \ldots, p_k\}$ is the set over which p ranges in the original construction. For b the modification is analogous, and for c add $q_I \xrightarrow{\sigma} \bot \diamond q_c$, where q_c is the initial state of the automaton recognizing $\left[L(A_{p_1}) \circ_1 L(A_{p'_1})\right] \sqcup \left[L(A_{p_2}) \circ_2 L(A_{p'_2})\right] \sqcup \cdots \sqcup \left[L(A_{p_k}) \circ_\ell L(A_{p'_\ell})\right]$, where $p_1 \circ_1 p'_1, p_2 \circ_2 p'_2, \ldots, p_\ell \circ_\ell p'_\ell$ are the triples over which $p \circ p'$ ranges in the original construction. Observe that these modifications do not influence the Wadge equivalence class of the recognized language. ◂

Thus we can assume that each non-trivial SCC of a given WGA is of one of the four forms presented in Fig. 5. By a Wadge game argument we can further simplify the automaton.

▸ **Lemma 15.** *For each* WGA *one can compute effectively a Wadge equivalent* WGA *over* $\{a, b, c\}$ *without non-trivial loops and branching transitions (except those for* $\top$, $\bot$*).*

After these simplifications we apply Lemma 9 to compute the Wadge degrees.

▸ **Theorem 16.** *For a given* WGA *one can effectively compute the Wadge equivalence class of the language it recognizes.*

Proof. By Lemma 15, we can assume that the automaton is over $\{a, b, c\}$, has no non-trivial loops, and no branching transitions. By induction on the DAG of SCCs we prove that the Wadge equivalence class of the recognized language is in $[\Omega]$ and can be computed effectively.

If the whole automaton consists of a single SCC, the result is $[1]^+$ or $[1]^-$ depending on the rank of the unique state.

To perform the inductive step, it suffices to express the recognized language in terms of the operations from Lemma 9. If there is no transition from the initial state q_I that leads back to q_I, the recognized language can be presented as $\left[L(A_{p_a}) \circ_a L(A_{p'_a})\right] \sqcup \left[L(A_{p_b}) \circ_b L(A_{p'_b})\right] \sqcup \left[L(A_{p_c}) \circ_c L(A_{p'_c})\right]$, where $q_I \xrightarrow{\sigma} p_\sigma \circ_\sigma p'_\sigma$ for $\sigma = a, b, c$.

For the rest of the proof we assume that the automaton is of the form shown in the leftmost part of Fig. 5; we use the notation $q_I(A, B, C)$. For the remaining possibilities the computations are analogous. If the $\mathsf{rank}\, q_I$ is even, consider the following cases.

1. $L(A) \geqslant_W \forall(L(B), \top)$. Then $L(A) >_W L(B)^{[n]}$ for every $n < \omega$, and we have that either $L(q_I(A, B, C)) \equiv_W L(q_I(A, B', C))$ for some B' recognizing a $\mathbf{\Sigma}^0_1$-complete language, if $d_W(L(B)) \geqslant [2]^-$, or $L(q(A, B, C)) \equiv_W L(q_I(A, \top, C))$ otherwise. In the former case the recognized language is Wadge equivalent to $\mathsf{loop\text{-}reset}^+(L(A), L(C))$, in the latter case it is Wadge equivalent to $\mathsf{loop}^+(L(A), L(C))$.
2. $L(A) <_W L(\forall(B, \top))$. The recognized language is Wadge equivalent to $L(q_I(\bot, B, C))$, which gives $\forall(L(B), L(C))$.
3. $L(A) \equiv_W L(\forall(B, \top))^\complement$. In this case, as $L(A) >_W L(B)^{[n]}$ for every $n < \omega$, we conclude that the recognized language is Wadge equivalent to $L(A) \diamond L(q(\bot, B, C)) \equiv_W L(A) \diamond \forall(L(B), L(C))$.

For $\mathsf{rank}\, q_I$ odd, dualize the above argument. ◂

7 Borel rank and weak index

As an immediate corollary of Theorem 16 we obtain decidability of the Borel rank problem.

▸ **Corollary 17.** *The problem of deciding the Borel rank of a* WGA*-recognizable language is decidable.*

We will now proceed to prove that the weak index conjecture holds for languages recognized by WGA. It has long been known that one implication holds.

▸ **Proposition 18** ([14])**.** Let $A \in$ WGA with index $(0, n)$ (resp. $(1, n+1)$). Then it holds that $L(A) \in \Pi^0_n$ (resp. $L(A) \in \Sigma^0_n$).

Using the connections between the structure and topological complexity of automata explained in the previous sections, we can prove the converse for WGA.

▸ **Theorem 19.** *For languages recognizable by* WGA*, the Borel hierarchy and the weak index hierarchy coincide.*

Proof. By duality and Proposition 18 it suffices to show that each WGA A recognizing a $\mathbf{\Pi}^0_n$ language admits an equivalent WATA of index $(0, n)$. We proceed by induction on the DAG of SCCs of the automaton.

If $n = 0$, A accepts every tree, so it is equivalent to a single state automaton of index $(0, 0)$. If $n = 1$, A cannot contain a productive state reachable from a nontrivial rejecting SCC, so an equivalent $(0, 1)$ automaton can be obtained by setting the rank of all states reachable from non-trivial rejecting SCCs to 1 and the rank of the remaining states to 0.

Suppose that $n \geqslant 2$, and let X be the root SCC. If X has rank 0 (we can change it to 0 if X is trivial), by Fact 5 (2) and the induction hypothesis we can present all A_q with $q \notin X$ as $(0, n)$ automata and the claim follows.

Suppose X is non-trivial and has rank 1. Assume that X contains a branching $\diamond$-transition. Then it follows that for all states q, $L(A_q)$ is in $\mathbf{\Sigma}^0_{n-1}$ (otherwise, the whole language would be $\mathbf{\Sigma}^0_n$ hard). In consequence, for all states $p \notin X$, A_p can be transformed into an equivalent WATA of index $(1, n)$, and we conclude like before.

The remaining case is that of non-trivial X of rank 1, without branching $\diamond$-transitions. Observe that in this case, there are two reasons why a tree can be rejecting:

1. a path of the computation stays forever in X, and for all $\diamond$ transitions in this path, the branches leaving X are rejecting;
2. a rejecting path exits X, and for all $\diamond$ transitions in this path, branches leaving X are rejecting.

By induction hypothesis, all A_p can be transformed to WATA of index $(1, n)$ if $q \xrightarrow{\sigma} p \diamond q'$ or $q \xrightarrow{\sigma} q' \diamond p$ for some $q, q' \in X$, or $(0, n)$ otherwise. To check that the second condition does not hold, use A with the rank X changed to 0. For the first condition, use A' obtained from A by replacing $q \xrightarrow{\sigma} p \circ p'$ with $q \xrightarrow{\sigma} \bot \diamond \bot$, $q \xrightarrow{\sigma} p \,\square\, q'$ with $q \xrightarrow{\sigma} \top \diamond q'$, and $q \xrightarrow{\sigma} q' \,\square\, p'$ with $q \xrightarrow{\sigma} q' \diamond \top$ for all $q, q' \in X$, $p, p' \notin X$. The ϵ-transition introduced to implement conjunction can be removed by unraveling the first step of the computation, without changing the ranks. ◂

This way the weak index problem reduces to the Borel rank problem. The construction above in fact gives an effective way of constructing the equivalent WATA of minimal index.

▸ **Corollary 20.** *The problem of calculating the exact position in the weak index hierarchy of a language recognized by a* WGA *is decidable and an equivalent* WATA *can be constructed effectively (in polynomial time if the productive states are given).*

8 Conclusions

We have isolated the class of game automata, a wide class of automata inducing operations on Wadge equivalence classes. For *weak* game automata we were able to use this property to describe all definable operations in terms of a small set of generators, and based on this we gave a procedure calculating the Wadge equivalence class of the language recognized by any given automaton. Using the structural information provided by the latter result we proved that the weak index hierarchy and the Borel hierarchy coincide, and gave algorithms computing the weak index and constructing an equivalent weak alternating automaton of the minimal index.

The results on the Wadge hierarchy subscribe to the line of reaserch aimed at investigating the hierarchies for families of languages recognized by various devices (cf. [5, 9, 21]). Usually, lower bounds on the heights of the hierarchies are easier to obtain, tight upper bounds are more difficult, and decidability results are scarce [7, 16, 24]. The peculiarity of this work is that we obtain computability of the Wadge degree without determining explicitly the inhabited levels of the hierarchy. Some lower bounds are easy to obtain based on our description of the induced operations and an upper bound is given by [Ω], but giving a full characterization of the inhabited levels seems to be a nontrivial task.

The class of automata we are considering has limited expressivity, but it seems to capture many interesting topological phenomena. Even more so in the unrestricted case, as game automata recognize the game languages recently considered by Arnold and Niwiński [2] in their study of the Wadge hierarchy of non-Borel regular languages. Currently, we are trying to drop the weakness restriction. One of the challenges is that for non-Borel languages the shape of Wadge hierarchy is unknown.

Despite the positive results concerning the hierarchy problems for weak game automata, and hopefully for non-weak, from the methodological point of view the message of this work is that we are reaching the limits of the topological approach to index problems. Pushing decidability results beyond game automata seems to require new techniques.

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References

1 A. Arnold, J. Duparc, F. Murlak, D. Niwiński. On the Topological Complexity of Tree Languages. In J. Flum et al. (Eds.) *Logic and Automata - History and Perspectives*, Texts in Logic and Games, Amsterdam University Press: 9–28 (2007).

2 A. Arnold, D. Niwiński. Continuous Separation of Game Languages. *Fund. Info.*, 81(1–3): 19–28 (2008).

3 J. Bradfield. The Modal μ-Calculus Alternation Hierarchy is Strict. *Theor. Comput. Sci.* 195(2): 133–153 (1998).

4 J. Duparc. Wadge Hierarchy and Veblen Hierarchy Part 1: Borel Sets of Finite Rank. *J. Symb. Log.* 66(1): 56–86 (2001).

5 J. Duparc. A Hierarchy of Deterministic Context-Free ω-Languages. *Theoret. Comput. Sci.* 290:1253–1300 (2003).

6 J. Duparc, F. Murlak. On the Topological Complexity of Weakly Recognizable Tree Languages. *FCT*: 261–273 (2007).

7 J. Duparc, A. Facchini, F. Murlak. Linear Game Automata: Decidable Hierarchy Problems for Stripped-Down Alternating Tree Automata. *CSL*: 225–239 (2009).

8 J. Duparc, A. Facchini, F. Murlak. Definable Operations On Weakly Recognizable Sets of Trees. `http://www.mimuw.edu.pl/~fmurlak/papers/gamafull.pdf`.

9 O. Finkel. Borel Ranks and Wadge Degrees of ω-Context Free Languages. *Mathematical Structures in Computer Science* 16: 813–840 (2006).

10 S. Hummel, H. Michalewski, D. Niwiński. On the Borel Inseparability of Game Tree Languages. *STACS*: 565–576 (2009).

11 A. S. Kechris. *Classical Descriptive Set Theory.* Graduate Texts in Mathematics, Vol. 156. Springer-Verlag, New York (1995).

12 L. H. Landweber. Decision Problems for ω-Automata. *Math. Systems Theory* 3: 376–384 (1969).

13 D. A. Martin. Borel Determinacy. *Ann. of Math.* (2), 102(2): 363–371 (1975).

14 A. W. Mostowski. Hierarchies of Weak Automata and Weak Monadic Formulas. *Theoret. Comput. Sci.* 83: 323–335 (1991).

15 F. Murlak. On Deciding Topological Classes of Deterministic Tree Languages. *CSL*: 573–584 (2005).

16 F. Murlak. The Wadge Hierarchy of Deterministic Tree Languages. *Logical Methods in Comput. Sci.*, 4(4), Paper 15.

17 F. Murlak. Weak Index vs Borel Rank. *STACS*: 573–584 (2008).

18 D. Niwiński, I. Walukiewicz. A Gap Property of Deterministic Tree Languages. *Theor. Comput. Sci.* 303: 215–231 (2003).

19 D. Niwiński, I. Walukiewicz. Deciding Nondeterministic Hierarchy of Deterministic Tree Automata. *Electr. Notes Theor. Comput. Sci.* 123: 195–208 (2005).

20 M. O. Rabin. Decidability of second-order theories and automata on infinite trees. Trans. Amer. Soc. **141**: 1–35 (1969).

21 V. Selivanov. Wadge Degrees of ω-Languages of Deterministic Turing Machines. *Theoret. Informatics Appl.*, 37: 67–83 (2003).

22 J. Skurczyński. The Borel Hierarchy is Infinite in the Class of Regular Sets of Trees. *Theoret. Comput. Sci.* 112: 413–418 (1993).

23 W. W. Wadge. *Reducibility and Determinateness on the Baire Space.* Ph.D. Thesis, Berkeley (1984).

24 K. Wagner. On ω-Regular Sets. *Inform. and Control* 43: 123–177 (1979).

A Alternating parity automata

An *alternating parity tree automaton* $\langle \Sigma, Q, q_I, \delta, \mathsf{rank} \rangle$ consists of a finite alphabet Σ, a finite set of states Q, an initial state $q_I \in Q$, a transition function $\delta : Q \times \Sigma \to \mathcal{B}^+(\{0,1\} \times Q)$, where $\mathcal{B}^+(\{0,1\} \times Q)$ denotes the set of positive boolean formulae over $\{0,1\} \times Q$, and a rank function $\mathsf{rank} : Q \to \mathbb{N}$. The acceptance is defined in terms of a max-parity game.
A *max-parity game* is a two-player game given by a graph $\langle V, V_\diamond, V_\Box, v_I, E, \mathsf{rank} \rangle$, where $v_I \in V$ is the initial position, $V = V_\diamond \cup V_\Box$ is the set of vertices, $E \subseteq V \times V$ is the edge relation, and $\mathsf{rank} : V \to \omega$ is a priority (or coloring) function, assuming only finitely many values. A vertex v is a successor of a vertex v' if $(v', v) \in E$ (we also write $v \in E(v')$). The players, $\diamond$ and $\Box$, play by moving a token, initially positioned in v_I, along the edges of the graph. If the token is in $v \in V_i$, the player i chooses the next location of the token among $E(v)$. A *play* is the path $v_0 v_1 v_2 \ldots$ with $v_0 = v_I$ that was taken by the token as a result of the players' moves.
If a player cannot make a move, he loses the play. In case of an infinite play, the player $\diamond$ wins if and only if the greatest priority occurring infinitely often in the sequence $\mathsf{rank}(v_0)\mathsf{rank}(v_1)\mathsf{rank}(v_2)\ldots$ is even.
Consider an alternating automaton A and a tree $t \in T_\Sigma$. The automaton A accepts t iff the player $\diamond$ has a winning strategy in the max-parity game $\langle V, V_\diamond, V_\Box, v_I, E, \mathsf{rank} \rangle$ defined as:

- $V = \mathcal{B}^+(\{0,1\} \times Q) \times \mathsf{dom}(t)$;
- $v_I = (\delta(q_I, t(\varepsilon)), \varepsilon)$.
- $V_\diamond$, $V_\Box$, E and rank are defined as follows: for each $(\psi, w) \in V$
  - if $\psi = \psi_1 \vee \psi_2$, then $(\psi, w) \in V_\diamond$, $E(\psi, w) = \{(\psi_1, w), (\psi_2, w)\}$, and $\mathsf{rank}(\psi, w) = 0$,
  - if $\psi = \psi_1 \wedge \psi_2$, then $(\psi, w) \in V_\Box$, $E(\psi, w) = \{(\psi_1, w), (\psi_2, w)\}$, and $\mathsf{rank}(\psi, w) = 0$,
  - if $\psi = (d, q)$, then $(\psi, w) \in V_\diamond$ (this is irrelevant), $E(\psi, w) = \{(\psi', wd)\}$ with $\psi' = \delta(q', t(wd))$, and $\mathsf{rank}(\psi, w) = \mathsf{rank}(q)$.

B Effective closure of ordinals: proof of Lemma 9

B.1 The Wadge hierarchy and the Wadge game

Recall that the space T_Σ of full binary trees over Σ equipped with the standard Cantor topology is a Polish space and is in fact homeomorphic to the Cantor space.
Consider $L \subseteq T_\Sigma$ and $M \subseteq T_{\Sigma'}$. We say that L *continously reduces* or *Wadge reduces* to M whenever there is a continous function $f : T_\Sigma \to T_{\Sigma'}$ such that $f^{-1}(L) = M$. If this is the case, "topologically" this means that L is less complicated than M and we write $L \leqslant_W M$. If $L \leqslant_W M$, but $M \nleqslant_W L$, then L is strictly less complicated than M, and we write $L <_W M$. Whenever $L \leqslant_W M$ and $M \leqslant_W L$, we say that L and M are *Wadge-equivalent*, and write $L \equiv_W M$.
The *Borel Wadge hierarchy* consists of the collection of all Borel tree languages ordered by the Wadge reduction.
The relation $\leqslant_W$ is clearly a pre-order. But more interestingly we have seen that it can be characterized by a two-player infinite game with imperfect information, called Wadge game and which is a special case of a Gale-Stewart game.
Let L and M be two arbitrary subsets of T_Σ. The Wadge game $\mathcal{W}(L, M)$ is played by two players, player I and player II. Both player build a tree, say t_I and t_{II}. At every round, player I plays first, and both players add a finite number of children to the terminal nodes of their corresponding tree. Player II is allowed to skip its turn, but not forever. We say that player II wins the game iff $t_I \in L \Leftrightarrow t_{II} \in M$. This game was designed precisely in order to obtain:

▸ **Lemma 21** ([23]). *Let $L, M \subseteq T_\Sigma$. Then $L \leqslant_W M$ iff player II has a winning strategy in the game $\mathcal{W}(L, M)$.*

Recall that a set L is called *self dual* if it is equivalent to its complement, otherwise it is called *non-self dual*. By the determinacy of Gale-Stewart games whose winning sets are Borel, if $L, M \subseteq T_\Sigma$ are Borel, then $\mathcal{W}(L, M)$ is determined. As a consequence, a straightfoward variant of Martin-Monk's result shows that $<_W$ is well-founded on Borel sets. Furthemore it is possible to prove that $\leqslant_W$-antichains have length at most two, and that the only incomparable Borel sets are – up to Wadge equivalence – of the form L and $L^\complement$, for L non-self-dual. This means that, up to complementation and Wadge equivalence, the Borel Wadge hierarchy is a well-ordering. Therefore, there exists a unique ordinal, called the height of the Borel Wadge hierarchy, and a mapping d_W from the Borel Wadge hierarchy onto its height, called the *Wadge degree*, such that $d_W(L) < d_W(M)$ if and only if $L <_W M$ and $d_W(L) = d_W(M)$ if and only if $L \equiv_W M$ or $L \equiv_W M^\complement$, for every Borel sets L and M. The wellfoundness of the Borel Wadge hierarchy ensures that the Wadge degree can be defined by induction as follows:

- $d_W(\varnothing) = d_W(\varnothing^\complement) = 1$,
- $d_W(L) = \sup\{d_W(K) + 1 \colon K \text{ non self dual}, K <_W L\}$ for $L >_W \varnothing$, non self-dual,
- $d_W(L) = \sup\{d_W(K) + 1 \colon K \text{ non self dual}, K <_W L\}$ for L self-dual.

If we consider only the class Δ^0_ω of Borel sets of finite rank, the height of the corresponding Wadge hierarchy is

$$\sup_{n\in\omega} \underbrace{\omega_1^{\omega_1^{\cdot^{\cdot^{\omega_1}}}}}_{n \text{ times}} = {}^{\omega_1}\epsilon_0,$$

the least fixpoint of the ordinal operation of exponentiation of base ω_1.

In [4], Duparc has shown how to construct for any $\alpha \in {}^{\omega_1}\epsilon_0$ a Borel set $\Omega(\alpha)$ of Wadge degree exactly α whose definition is isomorphic to the Cantor normal form of base ω_1 of the ordinal α. This can be done because of a remarkable correspondence between ordinal and set theoretical operations, a correspondence already (partially) discovered by Wadge [23]. More precisely, it is possible to associated to any of the following ordinal functions:

1. Ordinal sum: $(\alpha, \beta) \mapsto \alpha + \beta$,
2. Countable supremum: $(\alpha_n)_{n\in\omega} \mapsto \sup_{n\in\omega} \alpha_n$
3. Product: $(\alpha, \beta) \mapsto \alpha \cdot \beta$, for every $0 < \beta \leqslant \omega_1$,
4. Exponentiation of base ω_1: $\alpha \mapsto \omega_1^\alpha$

the following set theoretic operations:

1'. Sum: $(L, M) \mapsto L + M$,
2'. Countable supremum: $(L_n)_{n\in\omega} \mapsto \sum_{n\in\omega} L_n$
3'. Multiplication: $(L, \beta) \mapsto L \bullet \beta$, for every $0 < \beta \leqslant \omega_1$,
4'. Special action: $L \mapsto (L)^\sim$

which in addition have very natural interpretations in terms of Wadge games. We are thence able to generate the Wadge hierarchy of the sets in Δ^0_ω from scratch, that is starting from the empty set or its complement. As a consequence, we have that for every Wadge degree α, there is a canonical set $\Omega(\alpha)$ generated from the empty set only by the use of the previous set-theoretic operations such that for every set L of Wadge degree α, one of the three following possibility holds:

$$L \equiv_W \begin{cases} \Omega(\alpha) & \text{or} \\ \Omega(\alpha)^\complement & \text{or} \\ \Omega(\alpha) \sqcup \Omega(\alpha)^\complement. \end{cases}$$

Thus, we associate to every Wadge equivalence class A an unique *signed ordinal* $[\alpha]^\epsilon$, where

$$\epsilon = \begin{cases} - & \text{if } \Omega(\alpha) \in A \\ + & \text{if } \Omega(\alpha)^\complement \in A \\ \pm & \text{if } \Omega(\alpha) \sqcup \Omega(\alpha)^\complement \in A \end{cases}$$

From now on by a signed ordinal we denote at the same time the corresponding Wadge equivalence class, its canonical representative, but, by abusing notation, sometimes also the corresponding Wadge degree.

Let's have a closer look to some of the previous set-theoretic operations. Recall that a language is called *self dual* if it is equivalent to its complement. Since there is a trivial procedure which derives the structure of the self-dual classes from the non self-dual ones, we can concentrated on non self-dual sets only.

B.2 Set-Theoretical Operations

We start by recalling the four basic operations on sets of trees. Let $L, M \subseteq T_\Sigma$, and assume that Σ contains at least two letters, a and b. Define *alternative* ($\sqcup$), *parallel composition* ($\wedge$), *disjunctive product* ($\diamond$), and *conjunctive product* ($\square$) as

$$L \sqcup M = \{t \colon t(\varepsilon) = a\, , t.0 \in L \text{ and } t.1 \in M, \text{ or } t(\varepsilon) \neq a\}\, ,$$
$$L \diamond M = \{t \colon t.0 \in L \text{ or } t.1 \in M\}\, ,$$
$$L \square M = \{t \colon t.0 \in L \text{ and } t.1 \in M\}\, .$$

Multifold alternatives and parallel compositions are performed from left to right, e.g., $L_1 \sqcup L_2 \sqcup L_3 \sqcup L_4 = (((L_1 \sqcup L_2) \sqcup L_3) \sqcup A_4)$. It is easy to see that these four operations define associative and commutative operations on Wadge equivalence classes.
Another useuful operation on sets if the following. Let $L, M \subseteq T_\Sigma$. We define the set $L \to M$ as the set of trees $t \in T_{\Sigma \cup \{a\}}$, with $a \notin \Sigma$, satisfying any of the following conditions:

- $t.0 \in L$ and $a = t(10^n)$ for all n,
- 10^n is the first node on the path 10^* such that $a \neq t(10^n)$ and $t.10^n0 \in M$.

A player in charge of $L \to M$ is like a player in charge of L endowed with an extra move, which can be used only once, that erases everything played before. Then she can restart the play being in charge of M. We say that a non-self dual set $L \subseteq T_\Sigma$ is *initializable* when $L \geqslant_W L \to L$.

Sum and supremum : Suppose that $L, M \subseteq T_\Sigma$. We define the set $M + L$ as $L \to M \sqcup M^\complement$. From the point of view of the player in charge of the set $M + L$ in a Wadge Game, everything goes as if she was starting the game being in charge of L. So, provided she plays in such a way that a always holds in the rightmost branch of the tree, the question whether the resulting infinite tree she will have produced at the end of the run belongs to $M + L$ or not reduces to the question whether the tree starting from the left son of the root belongs to L or not. But at any moment of the run she can play a node 11^n not labelled with a. Then, everything looks like the whole (finite) tree played since the beginning of the game is erased and he is now in charge of: M if a is the label of the node (11^n1), $M^\complement$ else.
The following lemma ensures that the set-theoretical operation $+$ is well-behaved and in particular is the counterpart of the ordinal sum on Wadge degrees.

▸ **Lemma 22** ([5, 4])**.** *Let L, M, L', M' be four non self-dual Borel sets of binary trees. Then*

- $(L + M)^\complement \equiv_W L + M^\complement$,
- *The operation $+$ preserves the Wadge ordering:*

  if $L' \leqslant_W L$ and $M' \leqslant_W M$ then $L' + M' \leqslant_W L + M$

- $d_W(L + M) = d_W(L) + d_W(M)$.

As for alternative, it is easy to see that sum defines associative and commutative operations on Wadge equivalence classes.
The next operation is a generalization of $\sqcup$ and $+$. Let $\lambda < \omega_1$, and $L_\kappa \subseteq T_{\Sigma \cup \{b\}}$ for any $\kappa < \lambda$. Fix any $1-1$ map $f : \omega \to \lambda$. Thus, define $\sup^-_{\kappa<\lambda} L_\kappa$ as the set of trees $t \in T_{\Sigma \cup \{b\}}$ satisfying the following conditions for some k:

- 0^k is the first node on 1^* labeled with b,
- $t.0^k1 \in L_{f(k)}$.

Intuitively, a player in charge of $\sup^-_{\kappa<\lambda} L_\kappa$ is given the choice between the L_κ's. The decision is determined by the number of labels different from b played on the leftmost branch of the tree before the first b. If the player keeps not playing b forever on the leftmost branch, the tree will be rejected.
Define also $\sup^+_{\kappa<\lambda} L_\kappa$ as $\sup^-_{\kappa<\lambda} L_\kappa \cup \{t : \forall_n\ t(1^n) \neq b\}$. The difference from the previous operation is that now, when the player does not plays b on the leftmost branch, the obtained tree is accepted. Note that the operations are dual:

$$\left(\sup^+_{\kappa<\lambda} L_\kappa\right)^\complement = \sup^-_{\kappa<\lambda} \left(L_\kappa^\complement\right)$$

The following lemma ensures that the set-theoretical sup preserves the Wadge order and is the counterpart of the ordinal supremum on Wadge degrees.

▸ **Lemma 23** ([5, 4]). *Let $(L_\kappa)_{\kappa<\lambda}$ and $(M_\kappa)_{\kappa<\lambda}$ be two countable families of non self-dual Borel sets of binary trees. Then*
- *if for all $\kappa \in \lambda$, $L_\kappa \leqslant_W M_\kappa$ holds, then $\sup^+_{\kappa<\lambda} L_\kappa \leqslant_W \sup^+_{\kappa<\lambda} M_\kappa$ and $\sup^-_{\kappa<\lambda} L_\kappa \leqslant_W \sup^-_{\kappa<\lambda} M_\kappa$ hold too,*
- *$d_W(\sup^+_{\kappa<\lambda} L_\kappa) = [\sup_{\kappa<\lambda} d_W(L_\kappa)]^+$ and $d_W(\sup^-_{\kappa<\lambda} L_\kappa) = [\sup_{\kappa<\lambda} d_W(L_\kappa)]^-$.*

We state yet a simple observation. By $\bar{\mu}$ we denote the dual sign:

$$\bar{\mu} = \begin{cases} + & \text{if } \mu = -\,, \\ - & \text{if } \mu = +\,, \\ \pm & \text{if } \mu = \pm\,. \end{cases}$$

▸ **Lemma 24.** *For arbitrary Borel tree sets L_n,*

$$[1]^\mu \to \sup_n^\nu L_n \equiv_W \begin{cases} \sup_n^\nu L_n & \text{if } \mu = \nu \in \{-,+\} \\ [1]^\mu \to (\sup_n^+ L_n) \sqcup (\sup_n^- L_n) & \text{if } \mu = \bar{\nu} \in \{-,+\}\,. \end{cases}$$

Multiplication by ω_1: Let $L \subseteq T_\Sigma$. We define the set $L \bullet \omega_1$ as the set of trees $t \in T_{\Sigma\cup\{a,\overline{a}\}}$, with $a, \overline{a} \notin \Sigma$, satisfying one of the two following conditions:
- there is a n such that $a = t(0^n)$, $t.0^n1 \in L$ and $t(0^m) \in \Sigma$, for all $m > n$,
- there is a n such that $\overline{a} = t(0^n)$, $t.0^n1 \in L^\complement$ and $t(0^m) \in \Sigma$, for all $m > n$.

Note that if on the path 0^ω there are infinitely many nodes labeled by a or $\overline{a}$ then the tree is rejected.
The set-theoretical multiplication by ω_1 preserves the counterpart of the ordinal operation of multiplication by ω_1.

▸ **Lemma 25** ([5, 4]). *Let $L, M \subseteq T_\Sigma$ be two non self dual Borel sets. Then:*
- *if $L \leqslant_W M$ then $L \bullet \omega_1 \leqslant_W M \bullet \omega_1$*
- *$d_W(L \bullet \omega_1) = [d_W(L) \cdot \omega_1]^-$*

From the point of view of a player in charge of a set of the form $L \bullet \omega_1$, every position is equivalent to the initial one, meaning that a player in charge of $L \bullet \omega_1$ is like a player in charge of L with the additional possibility to restart the play at any moment being in charge of $L^\complement$ instead of L and start again and again replacing $L^\complement$ by L then L by $L^\complement$, and so and so forth, with the condition that a play that changes one's mind infinitely times is rejecting.
The dual operation is just define by adding the extra condition that if on the path 0^ω there are infinitely many nodes labeled by a or $\overline{a}$ then the tree is accepted. This operation, denoted by $\bullet\omega_1^+$ preserves the Wadge order and is such that $d_W(L \bullet \omega_1^+) = [d_W(L) \cdot \omega_1]^+$.

Special action : The next operation is a very peculiar but important one. It corresponds to the quasi-exponentiation of base ω_1. As this operation is quite complicated and its exact description is of no use for the present paper, we will just state some of its properties.

▸ **Lemma 26** ([4]). *Let L and M be two non-self dual Borel sets of trees. Then there is an operation $(\cdot)^\sim$ such that*
- *$((L)^\sim)^\complement \equiv_W (L^\complement)^\sim$,*
- *if $L \leqslant_W M$ then $(L)^\sim \leqslant_W (M)^\sim$*
- *$(L)^\sim \diamond (M)^\sim \equiv_W (L \diamond M)^\sim$.*

Moreover $d_W((L)^\sim) = \exp(d_W(L)+\epsilon)$ *where*

$$\epsilon = \begin{cases} -1 & \textit{if} \quad d_W(L) < \omega, \\ 0 & \textit{if} \quad d_W(L) = \beta + n \quad \textit{and} \quad \textit{cof}\beta = \omega_1, \\ +1 & \textit{if} \quad d_W(L) = \beta + n \quad \textit{and} \quad \textit{cof}\beta = \omega, \end{cases}$$

We have that:

▸ **Lemma 27** ([4])**.** *For every Borel tree languages L, L is inizializable iff there exists some ordinal α which is not of countable cofinality such that $d_W(L) = \omega_1^\alpha$.*

This means that $(L)^\sim$ and $L \bullet \omega_1$ are always initializable sets. Consistently with the fact above, recall that we call an ordinal $\beta < {}^{\omega_1}\epsilon_0$ initializable if there is some ordinal α which is not of countable cofinality such that $\beta = \omega_1^\alpha$.

As a conclusion of this subsection, let us now recall in one single proposition the elegant relation between operations on sets and Wadge degrees.

▸ **Proposition 28** ([4])**.** For arbitrary Borel sets L, M, L_n with $n < \omega$ it holds that

$$\begin{aligned} d_W(M+L) &= d_W(M) + d_W(L)\,, \\ d_W(M \sqcup L) &= \max(d_W(M), d_W(L))\,, \\ d_W(\overset{-}{\sup_n} L_n) = d_W(\overset{+}{\sup_n} L_n) &= \sup_n(d_W(L_n)+1)\,, \\ d_W(L \bullet \omega_1) = d_W(L \bullet \omega_1^+) &= d_W(L)\cdot\omega_1\,, \\ d_W((L)^\sim) &= \exp(d_W(L)+\epsilon)\,, \end{aligned}$$

$$\text{where } \epsilon = \begin{cases} -1 & \text{if } d_W(L) < \omega \\ 0 & \text{if } d_W(L) = \beta + n \text{ and } cof(\beta) = \omega_1 \\ +1 & \text{if } d_W(L) = \beta + n \text{ and } cof(\beta) = \omega. \end{cases}$$

B.3 Other basic properties of operations on signed ordinals

The following lemmas summarize simple yet useful properties of the operations on languages, and thence on signed ordinals (seen as Wadge equivalence classes of Borel sets). They can be proved with standard Wadge game arguments.

▸ **Lemma 29.** *For initializable A, B and arbitrary A', B' and A_n, B_n, $n < \omega$*

$$\begin{aligned}
(A \to A') \diamond (B \to B') &\equiv_W A \diamond B \to \big((A \to A') \diamond B' \sqcup A' \diamond (B \to B')\big)\ ,\\
(A \to A') \diamond B &\equiv_W A \diamond B \to A' \diamond B\ ,\\
(A \to A') \diamond \overset{+}{\sup_n} B_n &\equiv_W \overset{+}{\sup_n}\, (A \to A') \diamond B_n\ ,\\
(A \to A') \diamond \overset{-}{\sup_n} B_n &\equiv_W A \to \left((A' \diamond \overset{-}{\sup_n} B_n) \sqcup (\overset{-}{\sup_n}\, (A \to A') \diamond B_n) \sqcup (\overset{+}{\sup_n}\, (A \to A') \diamond B_n)\right) \quad \textit{for } A > \bot\ ,\\
(\bot \to A') \diamond \overset{-}{\sup_n} B_n &\equiv_W \bot \to \left((A' \diamond \overset{-}{\sup_n} B_n) \sqcup (\overset{-}{\sup_n}\, (\bot \to A') \diamond B_n)\right)\ ,\\
(\top \to A') \diamond \overset{-}{\sup_n} B_n &\equiv_W \top \to \left((A' \diamond \overset{-}{\sup_n} B_n) \sqcup (\overset{+}{\sup_n}\, (\top \to A') \diamond B_n)\right)\ ,\\
\overset{+}{\sup_m} A_m \diamond \overset{+}{\sup_n} B_n &\equiv_W \top \to \left(\overset{+}{\sup_m}(A_m \diamond \overset{+}{\sup_n} B_n) \sqcup (\overset{+}{\sup_n}\, (\overset{+}{\sup_m} A_m) \diamond B_n)\right)\ ,\\
\overset{+}{\sup_m} A_m \diamond \overset{-}{\sup_n} B_n &\equiv_W \top \to \left(\overset{+}{\sup_m}(A_m \diamond \overset{-}{\sup_n} B_n) \sqcup (\overset{+}{\sup_n}\, (\overset{+}{\sup_m} A_m) \diamond B_n)\right)\ ,\\
\overset{-}{\sup_m} A_m \diamond \overset{-}{\sup_n} B_n &\equiv_W \bot \to \left(\overset{-}{\sup_m}(A_m \diamond \overset{-}{\sup_n} B_n) \sqcup (\overset{-}{\sup_n}\, (\overset{-}{\sup_m} A_m) \diamond B_n)\right)\ .
\end{aligned}$$

▸ **Lemma 30.** *For arbitrary sets A, B, C, D it holds that*

- $(A \sqcup B) \diamond C \equiv_W A \diamond C \sqcup B \diamond C$,
- *if $A \leqslant_W C$ and $B \leqslant_W D$, then $A \diamond B \leqslant_W C \diamond D$.*

Finally, let us recall some well-known facts about the the difference hierarchy. For a Borel class Σ^0_n, the finite Hausdorff-Kuratowski, or difference, hierarchy is defined as $\mathrm{Diff}_1(\Sigma_n) = \Sigma_n$ and $\mathrm{Diff}_k(\Sigma_n) = \{U \backslash V : U \in \Sigma_n\ ,\ V \in \mathrm{Diff}_{k-1}(\Sigma_n)\}$. Let $\overline{\mathrm{Diff}_k(\Sigma_n)}$ denote the dual class. Recall that this is not the same as $\mathrm{Diff}_k(\Pi_n)$. Indeed, $\mathrm{Diff}_{2k+1}(\Pi_n) = \overline{\mathrm{Diff}_{2k+1}(\Sigma_n)}$ and $\mathrm{Diff}_{2k}(\Pi_n) = \mathrm{Diff}_{2k}(\Sigma_n)$. We have

$$\begin{aligned}
\mathrm{Diff}_{2k}(\Sigma_n) &= \{U_1 \cap V_1^{\complement} \cup \dots \cup U_k \cap V_k^{\complement}\}\ ,\\
\mathrm{Diff}_{2k+1}(\Sigma_n) &= \{U_1 \cap V_1^{\complement} \cup \dots \cup U_k \cap V_k^{\complement} \cup U\}\ ,\\
\overline{\mathrm{Diff}_{2k}(\Sigma_n)} &= \{U_1 \cap V_1^{\complement} \cup \dots \cup U_{k-1} \cap V_{k-1}^{\complement} \cup U \cup V^{\complement}\}\ ,\\
\overline{\mathrm{Diff}_{2k+1}(\Sigma_n)} &= \{U_1 \cap V_1^{\complement} \cup \dots \cup U_k \cap V_k^{\complement} \cup V^{\complement}\}\ ,
\end{aligned}$$

where the sets U, V, U_i, V_i range over Σ_n. From this characterization one easily obtains the following table of the operation $\diamond$. For $n > 0$ let $S_n(k)$ be a $\mathrm{Diff}_k(\Sigma_n)$-complete set, and let $P_n(k)$ be a $\overline{\mathrm{Diff}_k(\Sigma_n)}$-complete set.

▸ **Lemma 31.** *For each $n > 0$, $i > 0$, $j \geqslant 0$*

- $S_n(2i) \diamond S_n(2j) \equiv S_n(2i{+}2j)\ ,\ \ S_n(2i) \diamond P_n(2j) \equiv P_n(2i{+}2j)$
  $P_n(2i) \diamond S_n(2j) \equiv S_n(2i{+}2j)\ ,\ \ P_n(2i) \diamond P_n(2j) \equiv P_n(2i{+}2j{-}2)$
- $S_n(2i{+}1) \diamond S_n(2j) \equiv S_n(2i{+}2j{+}1)\ ,\ \ S_n(2i{+}1) \diamond P_n(2j) \equiv P_n(2i{+}2j)$
  $P_n(2i{+}1) \diamond S_n(2j) \equiv P_n(2i{+}2j{+}1)\ ,\ \ P_n(2i{+}1) \diamond P_n(2j) \equiv P_n(2i{+}2j)$
- $S_n(2i{+}1) \diamond S_n(2j{+}1) \equiv S_n(2i{+}2j{+}1)\ ,\ \ S_n(2i{+}1) \diamond P_n(2j{+}1) \equiv P_n(2i{+}2j{+}2)$
  $P_n(2i{+}1) \diamond S_n(2j{+}1) \equiv P_n(2i{+}2j{+}2)\ ,\ \ P_n(2i{+}1) \diamond P_n(2j{+}1) \equiv P_n(2i{+}2j{+}1)\,.$

We have already seen that:

▸ **Proposition 32** ([4])**.** For each $k > 0$, $d_W(S_n(k)) = d_W(P_n(k)) = \exp^n(k)$.

B.4 Nodes as operations on signed ordinals

We already now that some of the induced operations by nodes in a weak game automata have simple definitions in terms of ordinal arithmetics [7], and in particular they can be defined in terms of $\rightarrow$, $\diamond$ and the supremum operation. For instance

$$[\gamma_1]^{\epsilon_1} \sqcup [\gamma_2]^{\epsilon_2} = [\max(\gamma_1,\gamma_2)]^{\epsilon}, \text{ where } \epsilon = \begin{cases} \epsilon_1 & \text{if } \gamma_1 > \gamma_2 \\ \pm & \text{if } \gamma_1 = \gamma_2 \text{ and } \epsilon_1 \neq \epsilon_2, \\ \epsilon_2 & \text{otherwise} \end{cases}$$

$$\mathsf{loop}^+([\gamma_1]^{\epsilon_1},[\gamma_2]^{\epsilon_2}) = \Big[\sup_k \big(([\gamma_1]^{\epsilon_1})^{\langle k\rangle} \diamond [\gamma_2]^{\epsilon_2}\big)\Big]^+,$$

$$\forall([\gamma_1]^{\epsilon_1},[\gamma_2]^{\epsilon_2}) = [\exp^{i+1} 1]^+ \rightarrow \Big[\sup_k \big(([\gamma_1]^{\epsilon_1})^{[k]} \,\square\, [\gamma_2]^{\epsilon_2}\big)\Big]^+, \text{ for } [\exp^i 1]^+ \leqslant [\gamma]^\epsilon \leqslant [\exp^{i+1} 1]^-$$

$$[\exp^i(n)]^{\epsilon_1} \rightarrow [\gamma]^{\epsilon_2} = \begin{cases} [\gamma]^{\epsilon_2}, & \text{if } [\gamma]^{\epsilon_2} = [\beta + \exp^i(n)]^{\epsilon_1}, \\ [\gamma + \exp^i(n)]^{\epsilon_1}, & \text{else} \end{cases}$$

We now verify that also the two really new operations introduced by weak game automata (with respect to linear game automata), $\mathsf{loop\text{-}reset}^+$ and $\mathsf{loop\text{-}reset}^-$ shown in Fig. 3, can be defined in terms of known operations, namely in terms of $\rightarrow$, $\diamond$, $\square$, $\sup^-$, $\sup^+$, $\bullet\omega_1$ and $\bullet\omega_1^+$. More precisely:

▸ **Proposition 33.** Let L and M two non-self dual Borel sets. Then

$$\mathsf{loop\text{-}reset}^+(L,M) \equiv_W \begin{cases} [2]^- \rightarrow M & \text{if } L \equiv_W [1]^+ \\ (\sup^+_n(L^{\langle n\rangle})) \bullet \omega_1^+ \rightarrow \sup^+_n(L^{\langle n\rangle} \diamond M) & \text{otherwise} \end{cases},$$

$$\mathsf{loop\text{-}reset}^-(L,M) \equiv_W \begin{cases} [2]^+ \rightarrow M & \text{if } L \equiv_W [1]^- \\ (\sup^-_n(L^{\langle n\rangle})) \bullet \omega_1 \rightarrow \sup^-_n(L^{[n]} \,\square\, M) & \text{otherwise} \end{cases}.$$

Proof. It is enough to prove the first equivalence, the second following by duality.
The case when $L \equiv_W \emptyset^{\complement}$ is almost trivial. We therefore assume that $L >_W \emptyset^{\complement}$. Firstly, we describe a winning strategy for Player II in $\mathcal{W}(\mathsf{loop\text{-}reset}^+(L,M), (\sup^+_n(L^{\langle n\rangle})) \bullet \omega_1^+ \rightarrow \sup^+_n(L^{\langle n\rangle} \diamond M))$. There are three situations to describe.

1. Player I has not yet played any c on her rightmost path and: either he has not yet play any b on the rightmost path or for every node 0^n labelled by b, he plays accepting in the subtree $t.0^n1$. In this case Player II plays accepting by playing only a on her rightmost path,
2. Player I has played c at the node 0^m and for every node 0^n below 0^m labelled by b, he plays accepting in the subtree $t.0^n1$. In this case Player II erase everything she has played before and applies the winning strategy given by $L^{\langle \ell\rangle} \diamond M \leqslant_W \sup^+_k(L^{\langle k\rangle} \diamond M)$ with respect to what Player I is actually playing, where ℓ denotes the number of nodes labelled by a played by Player I below 0^m,
3. There is a least n such that Player I plays rejecting in the subtree $t.0^n1$, with $t(0^n) = b$. Thus Player II add a node, say 0^n, labelled by a to her rightmost branch, in the subtree $t.0^n1$ she applies the winning strategy given by $L^{\langle n-1\rangle} \leqslant_W \sup^+_k(L^{\langle k\rangle})$ with respect to what Player I has played below the node 0^n, and on her rightmost branch starts to play nodes not labelled in $\{a, \overline{a}\}$.

The strategy described by the three point above is clearly winning for Player II.
Finally, we describe a winning strategy for Player II in $\mathcal{W}((\sup^+_n(L^{\langle n\rangle})) \bullet \omega_1^+ \rightarrow \sup^+_n(L^{\langle n\rangle} \diamond M), \mathsf{loop\text{-}reset}^+(L,M))$. First notice that when L is of finite Wadge degree, $(\sup^+_n(L^{\langle n\rangle})) \bullet \omega_1^+ \equiv_W [\omega_1]^+$. In this case it is therefore very easy to determine a winning strategy for Player II. Thence, assume that $d_W(L)$ is uncountable. By Lemma 30, we have that if $n \leqslant m$, then $L^{\langle n\rangle} \leqslant_W L^{\langle m\rangle}$. Moreover since the degree of L is uncountable, $L^{\langle n\rangle} \rightarrow [1]^- \leqslant_W L^{\langle m\rangle} \rightarrow [1]^-$. Assume that Player I is still in charge of $\sup^+_n(L^{\langle n\rangle})) \bullet \omega_1^+$. Thus Player II has not yet played any c on her rightmost branch. Moreover, as long as Player

I plays accepting, Player II on her rightmost branch plays accepting on every left subtree of a node labeled by b and rejecting on every left subtree of a node labeled by a. Suppose that a certain point Player I chooses a k and decide to be in charge of $L^{\langle k\rangle}$. Thus Player II plays as follows:

- keeps playing rejecting in every subtree $t.0^k1$, with $t(0^k) = a$, played so far,
- plays a big enough number of new nodes labeled by a on her rightmost path, say $m > k$, and apply her winning strategy from $L^{\langle k\rangle} \to [1]^- \leqslant_W L^{\langle m\rangle} \to [1]^-$ in such a way that when Player II decides to erase all what he has played before and restart by being in charge of $\sup_n^+(L^{\langle n\rangle}))$, Player II can play rejecting on all the m considered subtrees, and start as before by playing accepting,
- after the new m nodes labeled by a, she plays a new node labeled by b, and in its left subtree she plays rejecting, switching to accepting as soon as Player I decides to erase everything he has played before and to restart playing by being in charge of $\sup_n^+(L^{\langle n\rangle}))$,
- mutatis mutandis, by using Lemmas 31 and 24, she acts analogously if Player I decides to restart by being in charge of $\sup_n^-(L^{[n]})$.

Reiterating this strategy as long as Player I keeps playing in $\sup_n^+(L^{\langle n\rangle})) \bullet \omega_1^+$ is winning for Player II. Thence, suppose that a certain point Player I decides to erase everything and to get in charge of $\sup_n^+(L^{\langle n\rangle} \diamond M)$. Thus, Player II has simply to

- play rejecting in every subtree $t.0^k1$, with $t(0^k) = a$,
- accepting in every subtree $t.0^k1$, with $t(0^k) = b$, she had played so far, and
- wait by playing only a on her rightmost branch until Player I has chosen the k for the set $L^{\langle k\rangle} \diamond M$ to be in charge of.

At this point she plays k new node labeled by a on her rightmost path, followed by a node labeled by c and apply the copy-cat winning strategy from $L^{\langle k\rangle} \diamond M \leqslant_W L^{\langle k\rangle} \diamond M$. Also this strategy is clearly winning for Player II. ◂

The previous result enables us to have an intuition behind the loop-reset operation. More precisely, a player in charge of $\mathsf{loop\text{-}reset}^+(L, M)$ is given first the possibility of choosing a finite n and then of being in charge $L^{\langle n\rangle}$ with the additional possibility to restart the play at any moment being in charge of $L^{\langle m\rangle}$, with $m \geqslant n$ instead of $L^{\langle n\rangle}$ and start again and again replacing $L^{\langle m\rangle}$ by $L^{\langle k\rangle}$, with $k \geqslant m$, and so and so forth, with the condition that a play that changes one's mind infinitely times is accepting. But the player in charge of $\mathsf{loop\text{-}reset}^+(L, M)$ has also the possibility when restarting the play, to decide of being in charge of the set $\sup_n^+(L^{\langle n\rangle} \diamond M)$. If she decides so, she cannot come back to a previous position.

In the next subsections we are finally going to prove that the class $[\Omega]$ is closed under the operations $\sqcup$, loop^+, $\mathsf{loop\text{-}reset}^+$, $\exists$ and the result of the operation can be computed effectively. Because of the previous result, this is done by showing that $[\Omega]$ is closed under the operations $\sqcup$, $\to$, $\sup^+$, $\sup^-$, $\sup^-(\cdot)\omega_1$ and $\diamond$.

B.5 Closure by $\sqcup$ and $\to$

The closure of $[\Omega]$ by $\sqcup$ is very simple. If $[\alpha]^\mu$ and $[\beta]^\nu$ are comparable, $[\alpha]^\mu \sqcup [\beta]^\nu$ is simply equal to the larger of the two. If $[\alpha]^\mu$ and $[\beta]^\nu$ are incomparable, then necessarily $[\beta]^\nu = [\alpha]^{\bar\mu}$ and the result is $[\alpha]^\pm$.

Let us now concentrate on $\to$. Observe that by Proposition 28 $[\alpha]^\mu \to [\beta]^\pm \equiv_W [\beta + \alpha]^\mu$. Thus, it follows that the result is in $[\Omega]$. Suppose $[\beta]^\nu = \sup^\nu[\beta' + \beta_n]^+$. If $\alpha \geqslant \omega$, we have $[\alpha]^\mu \to [\beta]^\nu \equiv_W ([\alpha]^\mu \to [1]^{\bar\nu}) \to [\beta]^\nu \equiv_W [\alpha]^\mu \to ([1]^{\bar\nu} \to [\beta]^\nu)$ By Lemma 24, this is equal to $[\alpha]^\mu \to [1]^{\bar\nu} \to [\beta]^\pm$. Hence, $[\alpha]^\mu \to [\beta]^\nu \equiv_W [\alpha]^\mu \to [\beta]^\pm$ and we conclude by the first case. For $\alpha < \omega$ use Lemma 24.

The remaining case is that $[\beta]^\nu = B \to [\beta']^\pm$, with B initializable. Suppose that $[\alpha]^+$ is not initializable. Then either $[\alpha]^+ <_W B$ or $[\alpha]^+ >_W B$. This implies that:

$$[\alpha]^+ \to B \equiv_W \begin{cases} [\alpha]^+ & \text{for} \quad B <_W [\alpha]^+ \\ B & \text{for} \quad B >_W [\alpha]^+\,, \end{cases}$$

and we conclude by inductive hypothesis. Consider now $[\alpha]^+ \to [\beta]^\nu$ for $\nu \in \{-,+\}$ with $[\alpha]^+$ initializable. It follows that

$[\alpha]^+ \to [\beta]^\nu \equiv_W ([\alpha]^+ \to B) \to [\beta']^\pm$. It is easy to see that

$$[\alpha]^+ \to B \equiv_W \begin{cases} [\alpha]^+ & \text{for} \quad B \leqslant_W [\alpha]^+ \\ B & \text{for} \quad B \geqslant_W [\alpha]^+ , \\ [\alpha \cdot 2]^+ & \text{for} \quad B \equiv_W [\alpha]^- , \end{cases}$$

and we can conclude from the previous case.

B.6 Closure by $\sup^+$, $\sup^-$, $\sup^-(\cdot)\omega_1$ and $\diamond$

In order to show the effective closure of $[\Omega]$ under the supremum but also under $\diamond$, we have first to determine properties of some special sequence of parametrized ordinals.

We call a sequence $(\alpha_n : n < \omega)$, with $\alpha_n < \omega^\omega$, *nice* if $\alpha_n \leqslant \alpha_{n+1}$ for every n and there is a m such that for every $k \geqslant m$:

- $\alpha_m = \alpha_k$, or
- α_k is the form $\beta + \omega^p \cdot k$ where $\beta < \omega^\omega$, $p < \omega$.

Thus, a sequence $([\alpha_n]^{\lambda_n} : n < \omega)$, with $[\alpha_n]^{\lambda_n} \in [\Omega]$ is called *well-behaved* if $[\alpha_n]^{\lambda_n} \leqslant_W [\alpha_{n+1}]^{\lambda_{n+1}}$ for every n and if there is a m such that for every $k \geqslant m$

(type A) $[\alpha_k]^{\lambda_k}$ is the form $[\gamma + \exp^\ell(k) + \epsilon]^\mu$, or

(type B) $[\alpha_k]^{\lambda_k}$ is the form $[\gamma + \exp^\ell(\eta)\iota_k + \epsilon]^\mu$, with $\mathrm{cof}(\eta) = \omega$ and $(\iota_k : k \geqslant m)$ is a nice sequence, or

(type C) $[\alpha_k]^{\lambda_k}$ is the form $[\gamma + \exp(\eta + \exp^\ell(k) + 1) + \epsilon]^\mu$, or

(type D) $[\alpha_k]^{\lambda_k}$ is the form $[\gamma + \exp(\eta + \exp^\ell(\omega)\iota_k + 1) + \epsilon]^\mu$ and $(\iota_k : k \geqslant m)$ is a nice sequence,

where $\epsilon \in \{0, 1\}$. If a well-behaved sequence is either of type A, C or D, is called *initializable*. Clearly $[\Omega]$ is closed under $\sup^+$ and $\sup^-$ of well-behaved sequences (initializable or not) and the result can be computed. We now verify that for every $[\alpha]^\mu, [\beta]^\nu \in [\Omega]$ we can always effectively extract from the sequence $(([\alpha]^\mu)^{\langle i \rangle} \diamond [\beta]^\nu : i < \omega)$ a well-behaved sequence with same limit, meaning that we can compute $\sup_i^+(([\alpha]^\mu)^{\langle i \rangle} \diamond [\beta]^\nu)$ and that the result is in $[\Omega]$, analogously for $\sup_i^-(([\alpha]^\mu)^{\langle i \rangle} \diamond [\beta]^\nu)$. From the proof of the closure of [7] under supremum, we already know the following:

▸ **Lemma 34** ([7]). *For every well-behaved sequence $([\alpha_n]^\mu \in [\Phi] : n < \omega)$ and every signed ordinal $[\beta]^\nu \in [\Phi]$, either the sequence $([\alpha_n]^\mu \diamond [\beta]^\nu \in [\Phi] : n < \omega)$ stabilizes after $n = 2$ or it is possible to compute a well-behaved sequence $([\kappa_n]^\lambda \in [\Phi] : n < \omega)$ whose limit is the same as the sequence $([\alpha_n]^\mu \diamond [\beta]^\nu \in [\Phi] : n < \omega)$.*

We now prove an analogous result for well-behaved sequences in $[\Omega]$. For a start, we prove it for initializable sequences. More precisely, we verify that:

▸ **Lemma 35.** *For every well-behaved initializable sequence $([\alpha_n]^- \in [\Omega] : n < \omega)$, with $\gamma = 0$, and every signed ordinal $[\beta]^\nu \in [\Omega]$, either the sequence $([\alpha_n]^- \diamond [\beta]^\nu \in [\Omega] : n < \omega)$ stabilizes after $n = 2$ or it is possible to compute a well-behaved sequence $([\kappa_n]^- \in [\Omega] : n < \omega)$ such that $\sup_n^\lambda [\kappa_n]^- = \sup_n^\lambda [\alpha_n]^- \diamond [\beta]^\nu \in [\Omega]$.*

Proof. We prove it by induction on β. We assume that $\epsilon = 0$, the other case being solved, mutatis mutandis, in the same way. As $[\alpha_k]^-$ is initializable, this means that

$$[\alpha_k]^- = \begin{cases} [\exp^\ell(k)]^- & \text{or} \\ [\exp(\eta + \exp^\ell(k) + 1)]^- & \text{or} \\ [\exp(\eta + \exp^\ell(\omega)\iota_k + 1)]^- \end{cases}$$

Since all three cases are solved in the same way, we just analyze the first one. Assume that β can be decomposed as $\beta_n.\omega^{i_n} + \cdots + \beta_0.\omega^{i_0}$, with $i_j < \omega$. If the sequence $([\alpha_n]^- \diamond [\beta]^\nu \in [\Omega] : n < \omega)$ stabilizes after $n = 2$ we are done. If this is not the case, this means that there is a greatest i such that for every k it holds that $[\beta_i]^\lambda \diamond [\alpha_k]^- >_W [\beta_i]^\lambda$, for $\lambda \in \{+,-\}$ if $i > 0$ and if $i = 0$ and $i_0 > 0$, $\lambda = \nu$ otherwise. Firstly, suppose that $[\beta_i]^\lambda = ([\eta_i]^\lambda)^\sim$. We have that the sequence $([\exp^{\ell-1}(k)]^- \diamond [\eta_i]^\lambda : k < \omega)$ is in $[\Phi]$ and by Lemma 34 is well-behaved. By Lemma 26, the sequence $([\sum_{j=n}^{i+1} \beta_j.\omega^{i_j} + \exp([\exp^{\ell-1}(k)]^- \diamond [\eta_i]^\lambda + \epsilon')]^\lambda : k < \omega)$ is also well-behaved and its supremum is in $[\Omega]$. Because there is $k' > k$ such that

$$[\sum_{j=n}^{i+1} \beta_j.\omega^{i_j} + \exp([\exp^{\ell-1}(k')]^- \diamond [\eta_i]^\lambda + \epsilon')]^\lambda \geqslant_W [\alpha_n]^- \diamond [\beta]^\nu \geqslant_W [\sum_{j=n}^{i+1} \beta_j.\omega^{i_j} + \exp([\exp^{\ell-1}(k)]^- \diamond [\eta_i]^\lambda + \epsilon')]^\lambda$$

we have that

$$\sup_k^\lambda [\sum_{j=n}^{i+1} \beta_j.\omega^{i_j} + \exp([\exp^{\ell-1}(k)]^- \diamond [\eta_i]^\lambda + \epsilon')]^\lambda = \sup_k^\lambda [\alpha_k]^- \diamond [\beta]^\nu,$$

meaning that $\sup_n^\lambda [\alpha_n]^- \diamond [\beta]^\nu \in [\Omega]$. The case when $[\beta_i]^\lambda$ is not initializable is obtained by using Lemma 29 and the induction hypothesis. ◂

From Lemma 35 we can already obtain that $[\Omega]$ is closed under disjunction. We first consider signed ordinals of the form $[\exp(\alpha)]^\mu$, for $\alpha \in \Phi$. If $[\exp(\alpha)]^\mu$ is not initializable, then this means that $\mathrm{cof}(\alpha) = \omega$ and therefore, as $\alpha \in \Phi$, that $\alpha = \gamma + \exp^n(\omega)\omega^p$, with $p, n < \omega$, and either $p \neq 0$ or $n \neq 0$. By Proposition 28, we have that either $\alpha = \sup_i(\gamma + \exp^n(i) + 1)$ or $\alpha = \sup_i(\gamma + \exp^n(\omega)\omega^{p-1} i + 1)$. We therefore have that if $[\exp(\alpha)]^\mu$ is not initializable we can compute a well-behaved sequence of ordinals $(\gamma(n) : n < \omega)$ such that by Lemma 29 for every $[\exp(\beta)]^\nu \in [\Omega]$ it holds that $[\exp(\alpha)]^\mu \diamond [\exp(\beta)]^\nu = \sup_n^\mu \gamma(n) \diamond [\exp(\beta)]^\nu$. By Lemma 35 $[\exp(\alpha)]^\mu \diamond [\exp(\beta)]^\nu$ is in $[\Omega]$ and can be computed.
Suppose now that both $[\exp(\alpha)]^\mu$ and $[\exp(\beta)]^\nu$ are initialializable. This implies that we know how to calculate $[\iota]^\mu, [\kappa]^\nu \in [\Phi]$ such that $([\iota]^\mu)^\sim = [\exp(\alpha)]^\mu$ and $([\kappa]^\nu)^\sim = [\exp(\beta)]^\nu$. By Lemma 26 we know that $[\exp(\alpha)]^\mu \diamond [\exp(\beta)]^\nu = ([\iota]^\mu \diamond [\kappa]^\nu)^\sim$. By the effective closure of $[\Phi]$ under $\diamond$, we conclude that $[\exp(\alpha)]^\mu \diamond [\exp(\beta)]^\nu \in [\Omega]$ and that it can be computed. We have thence obtained that:

▸ **Lemma 36.** *For each $[\alpha]^\mu, [\beta]^\nu \in [\Phi]$, one can effectively find $[\gamma]^\lambda \in [\Phi]$ such that $[\exp(\alpha)]^\mu \diamond [\exp(\beta)]^\nu = [\exp(\gamma)]^\lambda$.*

The effective closure of $[\Omega]$ under $\diamond$ follows thence simply by Lemmas 36 and 26 and inductive hypothesis.
In order to obtain the closure under supremum, we need an analogous of Lemmas 35 for some not initializable well-behaved sequences. In this aim, we need to have finer look at the computation of the disjunction operation when $[\alpha]^- = [\exp^\ell(\omega)\eta]^-$. First of all, we consider the case when $[\beta]^\nu = [\exp(\iota)\kappa]^\nu \in [\Omega]$. If $[\alpha]^- >_W [\beta]^\nu$ we have that $[\alpha]^- \diamond [\beta]^\nu = [\alpha + \beta]^\nu$. When $[\alpha]^- \leqslant_W [\beta]^\nu$, we distinguish two cases depending on whether $[\exp(\iota)]^\nu$ is initializable or not. Firstly, assume it is, meaning that $[\alpha]^- <_W [\beta]^\nu$. We have other two cases to consider: either $[\exp^\ell(\omega)]^- \diamond [\exp(\iota)]^- = [\exp(\iota)]^-$ or $[\exp^\ell(\omega)]^- \diamond [\exp(\iota)]^- >_W [\exp(\iota)]^-$. In the first case, $[\alpha]^- \diamond [\beta]^\nu = [\beta + \alpha]^-$. In the second case we obtain that $[\alpha]^- \diamond [\beta]^\nu = [\exp(\iota)(\kappa - 1)]^\nu \to [\gamma \cdot \eta]^-$, where

$$[\gamma]^- = \begin{cases} \sup_n^- ([\exp^\ell(n)]^- \diamond [\exp(\iota)]^+) \sqcup \sup_n^- ([\exp^\ell(n)]^- \diamond [\exp(\iota)]^-) & \text{if } \kappa > 0 \\ \sup_n^- ([\exp^\ell(n)]^- \diamond [\exp(\iota)]^\nu) & \text{otherwise.} \end{cases}$$

From Lemma 35 and the closure under sup, we have that $[\gamma \cdot \eta]^- \in [\Omega]$.
Assume that $[\exp(\iota)]^\nu$ is not initializable. We are only interested here in giving expressions for $\eta > 0$. Firstly, let $\nu = +$. Then we have that $[\alpha]^\mu \diamond [\beta]^\nu = [\exp^\ell(\omega)]^+ \diamond [\beta]^+ \sqcup [\exp^\ell(\omega)]^- \diamond [\beta]^+$. Consider now $\nu = -$. In this case, we have that $[\alpha]^\mu \diamond [\beta]^\nu = [\exp^\ell(\omega)(\eta - \epsilon)]^- \to \bigvee_{\lambda,\lambda' \in \{+,-\}} [\exp^\ell(\omega)]^\lambda \diamond [\beta]^{\lambda'}$, where $\epsilon = 0$ if $\mathrm{cof}(\eta) = \omega$, $\epsilon = 1$ otherwise.
The characterizations above are used for proving the next result.

▸ **Lemma 37.** *For every well-behaved sequence* $([\alpha_n]^- \in [\Omega] : n < \omega)$ *which is not initializable, with* $\gamma = 0$ *and* $\eta = \omega$, *and for every signed ordinal* $[\beta]^\nu \in [\Omega]$, *either the sequence* $([\alpha_n]^- \diamond [\beta]^\nu \in [\Omega] : n < \omega)$ *stabilizes after* $n = 2$ *or it is possible to compute a well-behaved sequence* $([\kappa_n]^- \in [\Omega] : n < \omega)$ *such that* $\sup_n^\lambda [\kappa_n]^- = \sup_n^\lambda [\alpha_n]^- \diamond [\beta]^\nu \in [\Omega]$.

Proof. We prove it by induction on β. Every $[\alpha_k]^-$ is of the form $[\exp^\ell(\omega)\iota_k]^-$ and $(\iota_k : k \geqslant m)$ is a nice sequence. Assume $\beta = \sum_{i=N}^1 \beta_i + \exp^\ell(\omega)\iota' + \beta_0$, with $\beta_N \geqslant \cdots \geqslant \beta_2 \geqslant \beta_1 > \exp^\ell(\omega) > \beta_0$, and each $\beta_i = \exp(\eta_i)\kappa_i$, where $\kappa = 1$ if $\exp(\eta)$ initializable, $\kappa < \omega^\omega$ otherwise. Firstly, let $\exp^\ell(\omega)\iota' + \beta_0 > 0$. Suppose there is a greatest $i = 1, \cdots, N$ such that $\exp(\eta_i)$ is initializable and $[\alpha_k]^- \diamond [\exp(\eta_i)]^- >_W [\exp(\eta_i)]^-$. This means that:

$$[\alpha_k]^- \diamond [\beta]^\nu = [\sum_{j=i+1}^{N} \beta_j + \gamma \cdot \iota_k + \sum_{j=1}^{i-1} \beta_j + \exp^\ell(\omega)\iota' + \beta_0]^\nu$$

where $\gamma^- = [\exp^\ell(\omega)]^- \diamond [\exp(\eta_i)]^+ \sqcup [\exp^\ell(\omega)]^- \diamond [\exp(\eta_i)]^-$. Let $\kappa_k = \sum_{j=i+1}^N \beta_j + \gamma \cdot \iota_k$. Since $(\iota_k : k < \omega)$ is a nice sequence, we have that $([\kappa_k]^- : k < \omega)$ is well-behaved. As $[\kappa_{k+1}]^- \geqslant_W [\alpha_k]^- \diamond [\beta]^\nu \geqslant_W [\kappa_k]^-$ we obtain that $\sup_n^\lambda [\kappa_n]^- = \sup_n^\lambda [\alpha_n]^- \diamond [\beta]^\nu \in [\Omega]$. Suppose now that there is no i such that $\exp(\eta_i)$ is initializable and $[\alpha_k]^- \diamond [\exp(\eta_i)]^- >_W [\exp(\eta_i)]^-$. Thence, consider the sequence

$$([\exp^\ell(\omega)\iota_k^* + 1]^\mu \to \bigvee_{\lambda,\lambda' \in \{+,-\}} \beta'^\lambda \diamond \exp^\ell(\omega)^{\lambda'} : k < \omega)$$

where $\beta' = \sum_{i=N}^1 \beta_i$ and $\begin{cases} [\exp^\ell(\omega)\iota_k^*]^- = [\exp^\ell(\omega)\iota_k]^- \diamond [\exp^\ell(\omega)\iota']^- & \text{if } \beta_0, \exp^\ell(\omega)\iota' > 0 \\ [\exp^\ell(\omega)\iota_k^*]^- = [\exp^\ell(\omega)\iota_k]^- & \text{if } \beta_0 > 0 \text{ but } \exp^\ell(\omega)\iota' = 0 \\ [\exp^\ell(\omega)\iota_k^*]^\nu = [\exp^\ell(\omega)\iota_k]^- \diamond [\exp^\ell(\omega)\iota']^\nu & \text{otherwise.} \end{cases}$

This sequence either stabilizes after $n = 2$ or is well-behaved. As

$$[\exp^\ell(\omega)\iota_{k+1}^* + 1]^\mu \to \bigvee_{\lambda,\lambda' \in \{+,-\}} \beta'^\lambda \diamond \exp^\ell(\omega)^{\lambda'} \geqslant_W [\alpha_k]^\mu \diamond [\beta]^\nu \geqslant_W [\exp^\ell(\omega)\iota_k^* + 1]^\mu \to \bigvee_{\lambda,\lambda' \in \{+,-\}} \beta'^\lambda \diamond \exp^\ell(\omega)^{\lambda'}$$

the claim follows. Secondly, let $\exp^\ell(\omega)\iota' + \beta_0 = 0$.
Suppose there is a greatest i such that $\exp(\eta_i)$ is initializable and $[\alpha_k]^- \diamond [\exp(\eta_i)]^- >_W [\exp(\eta_i)]^-$, and consider the sequence $([\beta' + \gamma \cdot \iota_k]^- : k < \omega)$ where $\beta' = \sum_{i=N}^1 \beta_i$ and

$$[\gamma]^- = \begin{cases} \sup_n^-([\exp^\ell(n)]^- \diamond [\exp(\eta_i)]^+) \sqcup \sup_n^-([\exp^\ell(n)]^- \diamond [\exp(\eta_i)]^-) & \text{if } i > 1 \\ \sup_n^-([\exp^\ell(n)]^- \diamond [\exp(\iota)]^\nu) & \text{otherwise.} \end{cases}$$

This sequence is well-behaved and there is a computable k', depending on the number of $j \neq i$ with $\beta_j = \exp(\eta_i)$, such that

$$[\beta' + \gamma \cdot \iota_{k+k'+2}]^- \geqslant_W [\alpha_k]^- \diamond [\beta]^\nu \geqslant_W [\beta' + \gamma \cdot \iota_{k+k'}]^-$$

meaning that $\sup_k^- [\alpha_k]^- \diamond [\beta]^\nu$ is in $[\Omega]$ and can be computed.
Suppose there is no i such that $\exp(\eta_i)$ is initializable and $[\alpha_k]^- \diamond [\exp(\eta_i)]^- >_W [\exp(\eta_i)]^-$. If $\nu = +$, then $\bigvee_{\lambda \in \{+,-\}} \beta^+ \diamond \exp^\ell(\omega)^\lambda$ and the claim is proved. Otherwise, as before, consider the sequence

$$([\exp^\ell(\omega)(\iota_k - \epsilon) + 1]^\mu \to \bigvee_{\lambda \in \{+,-\}} \beta^\nu \diamond \exp^\ell(\omega)^\lambda : k < \omega),$$

with $\epsilon = 0$ if $\mathrm{cof}(\iota_k) = \omega$, $\epsilon = 0$ otherwise. This sequence either stabilizes after $n = 2$ or is well-behaved and we know it has same limit as the sequence $([\alpha_k]^\mu \diamond [\beta]^\nu : k < \omega)$. ◂

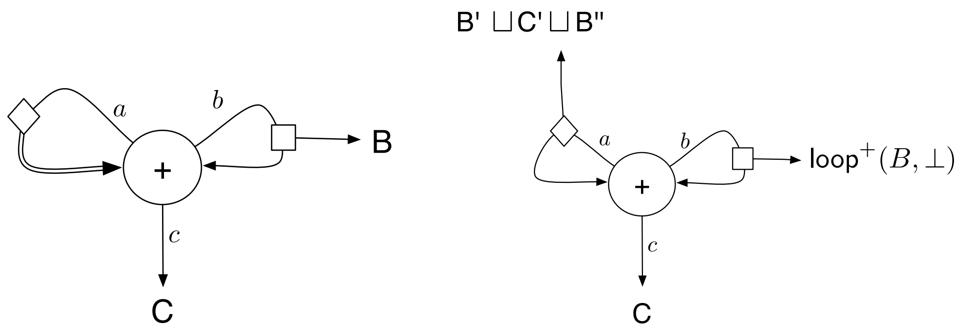
Everything now is ready to complete the proof of the closure properties of $[\Omega]$. By using the Lemmas 35 and 37, we verify that for every $[\alpha]^\mu, [\beta]^\nu \in [\Omega]$ we can always compute $\sup_i^+(([\alpha]^\mu)^{\langle i \rangle} \diamond [\beta]^\nu)$ and that the result is in $[\Omega]$, analogously for $\sup_i^-(([\alpha]^\mu)^{\langle i \rangle} \diamond [\beta]^\nu)$.
We start by considering $(([\alpha]^\mu)^{\langle i \rangle} : i < \omega)$. First, if $([\alpha]^\mu)^{\langle 2 \rangle} = [\alpha]^\mu$ the result is trivial and $\sup_i^+(([\alpha]^\mu)^{\langle i \rangle} \diamond [\beta]^\nu)$ is in $[\Omega]$ and can be computed. If $([\alpha]^\mu)^{\langle 2 \rangle} >_W [\alpha]^\mu$, we reason case by case as follows.

1. Assume that $[\alpha]^\mu = [\exp^\ell(r)k + \kappa]^\mu$. Define $[\gamma_n]^\mu = [\exp^\ell(n) + \epsilon]^\mu$
2. Assume that $[\alpha]^\mu = [\exp^\ell(\omega)\omega^p + \kappa]^\mu$. Define $[\gamma_n]^\mu = [\exp^\ell(\omega)\omega^p n + \epsilon]^\mu$
3. Assume that $[\alpha]^\mu = [\exp(\iota)\omega^p + \kappa]^\mu$, where $\iota = \exp^\ell(r)k + \kappa'$. Define $[\gamma_n]^\mu = [\exp(\exp^\ell(n) + 1) + \epsilon]^\mu$
4. Assume that $[\alpha]^\mu = [\exp(\iota)\omega^p + \kappa]^\mu$, where $\iota = \exp^\ell(\omega)\omega^q + \kappa'$. Define $[\gamma_n]^\mu = [\exp(\exp^\ell(\omega)\omega^q n + 1) + \epsilon]^\mu$,

where $\epsilon = 0$ if $\kappa = 0$, and $\epsilon = 1$ otherwise. For all cases above, $([\gamma_n]^\mu : n < \omega)$ is well-behaved and it is easy to check that there exists a $k > 0$ such that for every n: $[\gamma_{kn+2}]^\mu \geqslant_W ([\alpha]^\mu)^{\langle n \rangle} \geqslant_W [\gamma_{kn}]^\mu$. From Lemma 35 $\sup^\lambda_n [\alpha_n]^\mu \diamond [\beta]^\nu$ can be computed and is in $[\Omega]$.
From what precede we also immediately obtain that $[\Omega]$ is closed under $\sup^-(\cdot)\omega_1$.

C Simpifying automata: proof of Lemma 15

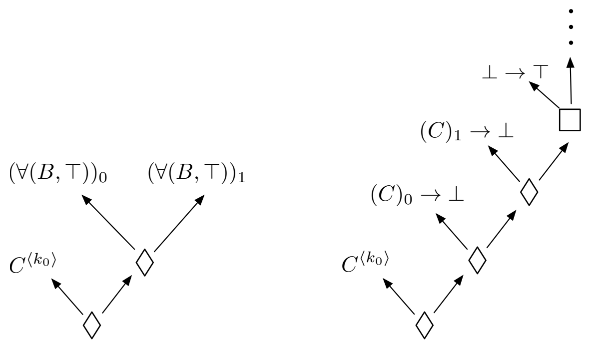
Thanks to Lemma 14, we know that for each WGA one can effectively compute a Wadge equivalent WGA over $\{a, b, c\}$ without non-trivial loops. We are now going to proof that this reduction can be improved. Namely, we prove that for every WGA over $\{a, b, c\}$ without non-trivial loops, one one can compute effectively a Wadge equivalent WGA over $\{a, b, c\}$ without non-trivial loops and branching transitions (except those for $\top$, $\bot$). Moreover it can be verified that such a reduction preserves the index.
Consider the weak game automaton D over $\{a, b, c\}$ without non-trivial loops in Figure 6a. We assume that the subautomata A, B, C have neither non-trivial loops nor branching transitions. We show that D is Wadge equivalent to the automaton D' without non-trivial loops and branching transitions of Figure 6b, where $B' = \forall(B, \top)$, $B'' = \forall(B, C')$ and $C' = \mathsf{loop}^+(C, \bot)$.



(a) The automaton D **(b)** The reduced automaton D'

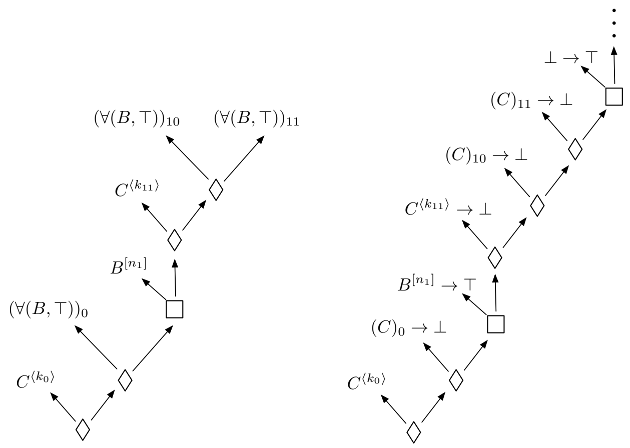
Figure 6 Two Wadge equivalent weak game automata

First lemma that if $L(B)$ is $\mathbf{\Pi}^0_n$-complete and $L(C)$ is $\mathbf{\Sigma}^0_n$-complete, then $L(D)$ is $\mathrm{D}_2(\mathbf{\Pi}^0_n)$-complete and that if $L(B') \geqslant_W L(C)$ then $L(B') \equiv_W L(D)$. Thus, w.l.o.g., we can assume that $L(C) >_W L(B') >_W [2]^+$. We first show that $L(D) \leqslant_W L(D')$. Roughly, the argument goes as follows. We can see a position for Player I in the Wadge game $\mathcal{W}(L(D), L(D'))$ as a finite tree where every internal node is labeled with either $\diamond$ or $\square$, and every leaf is labeled either with $C^{\langle k \rangle}$, or $B^{[k]}$ or $\forall(B, \top)$. Such a tree denotes the actual position of the run of the automaton D over the finite tree played by Player I in the original Wadge game. Analyzing the class of possible runs of D, we prove the existence of a winning strategy for Player II.
Suppose that the finite tree played by Player I induces the run of D depicted in Figure 7a. Then Player II plays the finite tree inducing the run of D' as depicted in Figure 7b, where to the subtree denoted by $(C)_i \to \bot$, Player II applies the winning strategy given by the fact that $L(\forall(B, \top) \leqslant_W L(C)$ with respect to the subtree denoted by $(\forall(B, \top))_i$, for $i = 0, 1$, in $(\bot \to \top)$ he simply plays rejecting, and in all the other subtrees he applies the copy-cat winning strategy. Now, suppose that at a certain point Player I stops to play into the subtree denoted by, say, $(\forall(B, \top))_1$. I.e., he is actually playing in the tree of Figure 8a
Then Player II "kills" the subtree $(\bot \to \top)$ by playing accepting, and he thence answers with the tree in Figure 8b Assume now that Player I also stops to play into $L(\forall(B, \top))$ in the subtree denoted by $(\forall(B, \top))_0$, and he thus plays in the tree of Figure 9a.



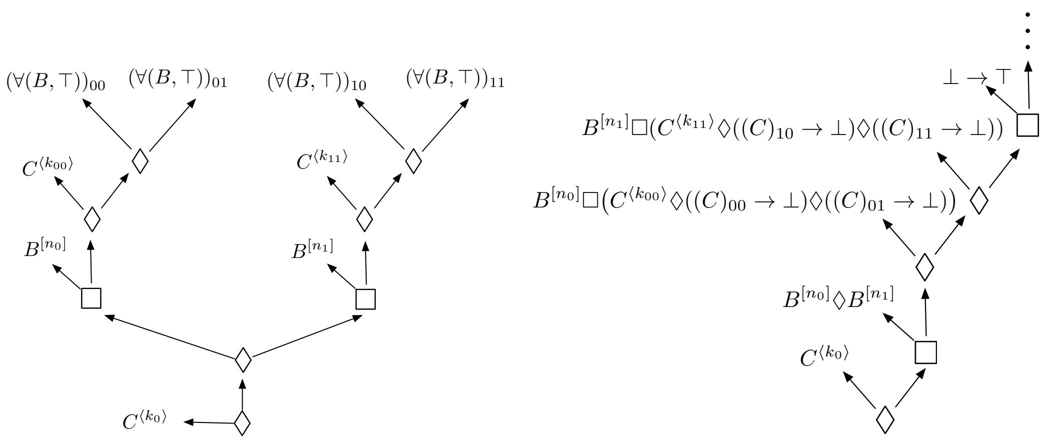
(a) Player I in charge of $L(D)$ **(b)** Player II in charge of $L(D')$

Figure 7 A w.s. for Player I in $\mathcal{W}(L(D),L(D'))$ (step 1)



(a) Player I in charge of $L(D)$ **(b)** Player II in charge of $L(D')$

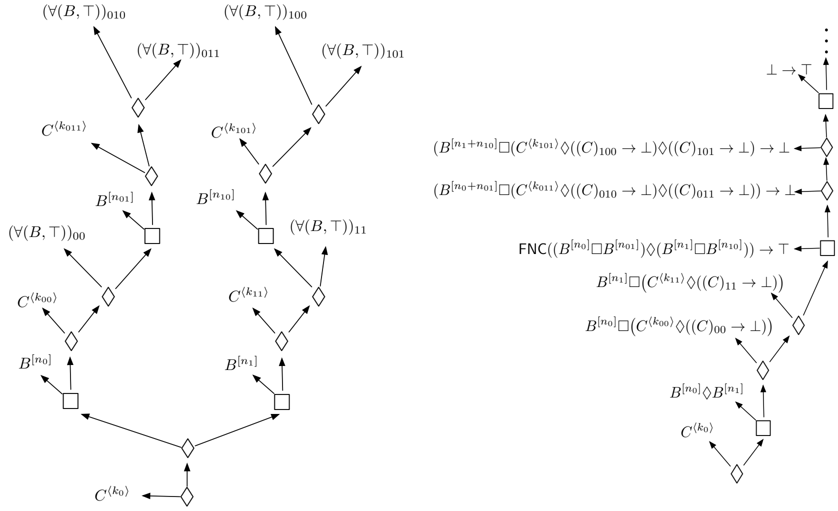
Figure 8 A w.s. for Player I in $\mathcal{W}(L(D),L(D'))$ (step 2)



(a) Player I in charge of $L(D)$ **(b)** Player II in charge of $L(D')$

Figure 9 A w.s. for Player I in $\mathcal{W}(L(D),L(D'))$ (step 3)

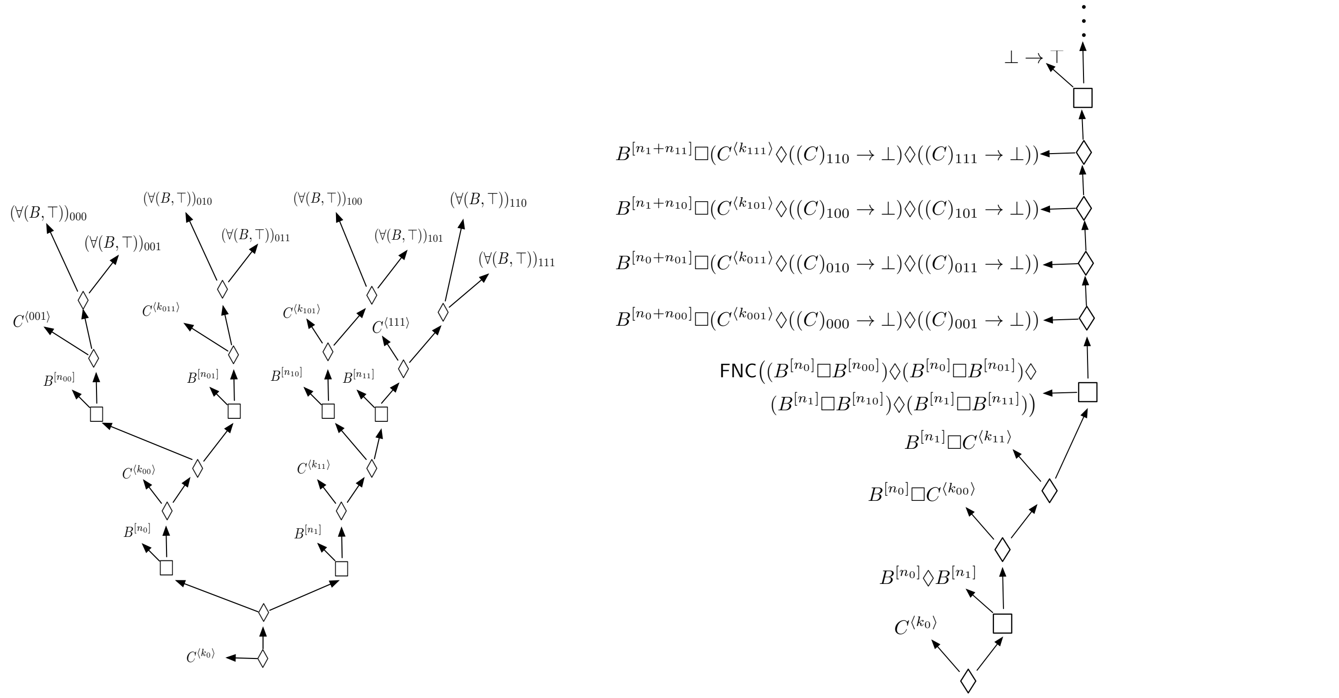
Then by playing either rejecting or accepting, Player II "kills" all the subtrees except for the one denoted by $C^{\langle k_0 \rangle}$, and he answers with the tree of Figure 9b, where to the subtree denoted by $(C)_i \to \bot)$, Player II applies the winning strategy given by the fact that $L(\forall(B, \top) \leqslant_W L(C)$ with respect to the subtree denoted by $(\forall(B, \top))_i$,in $(\bot \to \top)$ he simply plays rejecting, and in all the other subtrees he applies the copy-cat winning strategy.
Assume now that Player I stops to play into $L(\forall(B, \top))$ in the subtrees denoted by $(\forall(B, \top))_{01}$ and by $(\forall(B, \top))_{10}$, and he thus plays in the tree of figure 10a.



(a) Player I in charge of $L(D)$ **(b)** Player II in charge of $L(D')$

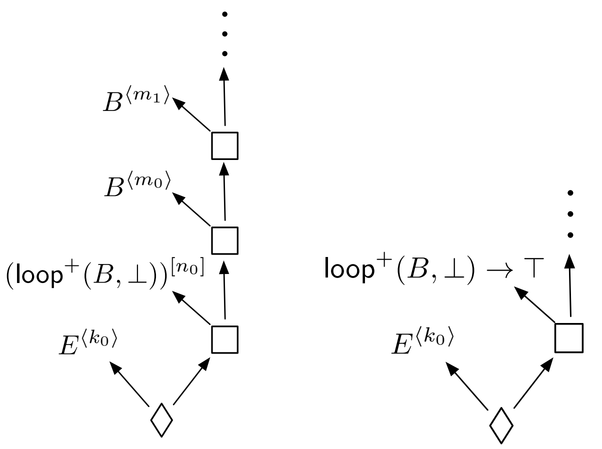
Figure 10 A w.s. for Player I in $\mathcal{W}(L(D), L(D'))$ (step 4)

Then Player II answers with the tree in Figure 10b, where $\mathsf{FNC}((B^{[n_0]} \,\Box\, B^{[n_{01}]}) \diamond (B^{[n_0]} \,\Box\, B^{[n_{10}]}))$ denotes the equivalent conjunctive normal form. Assume now that Player I stops to play into $L(\forall(B, \top))$ also in the subtrees denoted by $(\forall(B, \top))_{00}$ and by $(\forall(B, \top))_{11}$, and he thus plays in the tree of Figure 11a.
Then Player II answers with the tree of Figure 11b. It is therefore clear that following this kind of strategy is winning for Player II.
For the other direction we reason as follows. Suppose that the finite tree played by Player I induces the following run of D' is the one depicted in Figure 12a, where $E = B' \sqcup C' \sqcup B''$.
Then Player II plays the following finite tree inducing the tree induced by the run of D as depicted in Figure 12b, where to the subtree denoted by $\mathsf{loop}^+(B, \bot) \to \top$, Player II applies the winning strategy given by the fact that $L(\forall(\mathsf{loop}^+(B, \bot)), \top) \leqslant_W L(\mathsf{loop}^+(B, \bot))$ with respect to the subtree starting at the first node labeled by $\Box$ on the rightmost path of the tree induced by Player I. In the left subtree he just applies the winning copy-cat strategy.
Now, suppose that at a certain point Player I reaches the position in Figure 13a.
Then Player II "kills" the subtree $\mathsf{loop}^+(B, \bot) \to \top$ by playing accepting, and he thence answers with the tree of Figure 13b, where $B^{[m'_0]} \diamond \cdots \diamond B^{[m'_j]} = \mathsf{FND}(B^{\langle m_0 \rangle} \,\Box\, B^{\langle m_0 \rangle})$.
Assume now that Player I also stops to play trivially accepting in $L((\mathsf{loop}^+(B, \bot))^{[n_0]})$, and therefore reaches the position of Figure 14a.
Then Player II "kills" all the the subtrees except for the one denoted by $C^{\langle k_0 \rangle}$, and he answers with the tree depicted in Figure 14b, where $B^{[m''_0]} \diamond \cdots \diamond B^{[m''_j]} = \mathsf{FND}(B^{\langle m_0 \rangle} \,\Box \cdots \Box\, B^{\langle m_i \rangle})$. Clearly if Player I does not plays C on the rightmost path of his induced tree, following the previous strategy is winning for Player II.
Thus, assume now that Player I decides to play C on the rightmost path of the tree of Figure 15a.
Then Player II has simply to answers with the tree of Figure 15b, and eventually apply the previous modification of the strategy when Player I decides to stop playing trivially accepting in $L((\mathsf{loop}^+(B, \bot))^{[n_1]})$.
The depicted strategy for Player II is clearly winning.



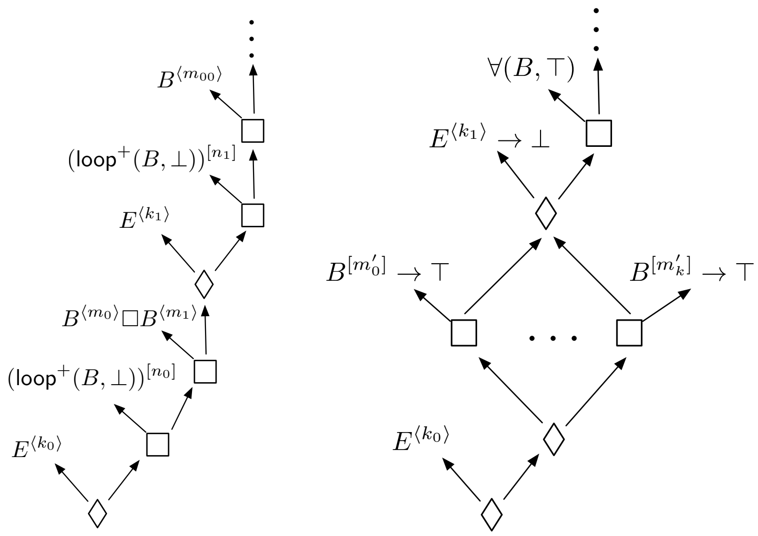
(a) Player I in charge of $L(D)$ **(b)** Player II in charge of $L(D')$

Figure 11 A w.s. for Player I in $\mathcal{W}(L(D), L(D'))$ (step 5)



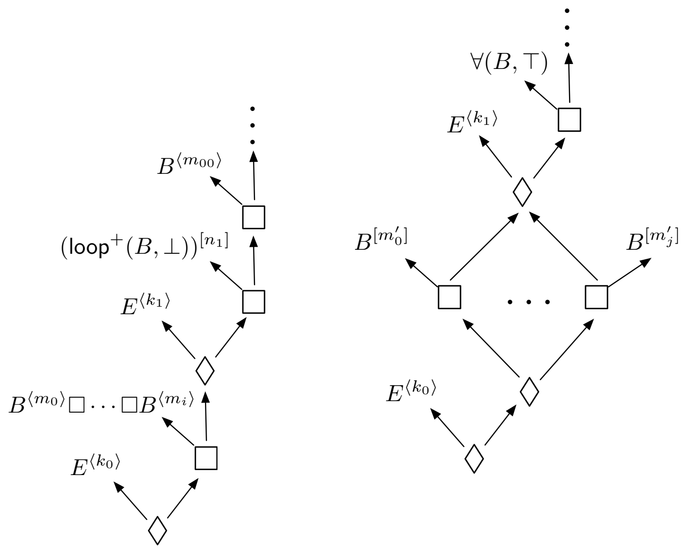
(a) Player I in charge of $L(D')$ **(b)** Player II in charge of $L(D)$

Figure 12 A w.s. for Player I in $\mathcal{W}(L(D'), L(D))$ (step 1)



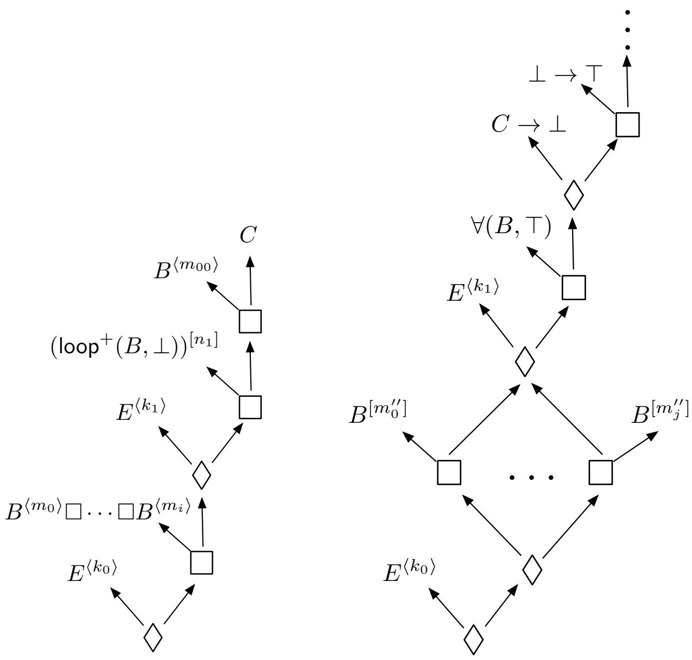
(a) Player I in charge of $L(D')$ **(b)** Player II in charge of $L(D)$

Figure 13 A w.s. for Player I in $\mathcal{W}(L(D'), L(D))$ (step 2)



(a) Player I in charge of $L(D')$ **(b)** Player II in charge of $L(D)$

Figure 14 A w.s. for Player I in $\mathcal{W}(L(D'), L(D))$ (step 3)



(a) Player I in charge of $L(D')$ **(b)** Player II in charge of $L(D)$

Figure 15 A w.s. for Player I in $\mathcal{W}(L(D'), L(D))$ (step 4)