On Ontological Functors of Leśniewski's Ontology

by

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Abstract In our work we elaborate on some problems connected with functors of Leśniewski's systems, especially with the functors of Ontology. We begin with discussing the simplest case — functors of Protothetic and their definability. Next, we present an algorithm which allows to define any possible sentence-formative functor of Leśniewski's Elementary Ontology(LEO), arguments of which belong to the category of names (other results of this chapter are: a recursive method of listing possible functors, a method of indicating the number of possible n-place ontological functors, and a sketch of a proof that LEO is functionally complete with respect to $\{\wedge, \neg, \forall, \varepsilon\}$). Next, we apply some of our apparatus to the problem of adequacy of axioms of Leśniewski's Ontology to the usually given semantic characteristics of the functor ε. Later, we discuss the distinction between mereological and distributive understanding of the functors: Kl and el, and by evaluating the role of this distinction in Leśniewski's solution to the Russell's paradox. Finally, we end our work by devoting some time to investigations of historical nature — we construct a name calculus without quantification and use it to answer the question, whether Syllogistic was, and to what extent it can be, treated as a free logic.

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1 Introduction

We shall proceed according to the following plan:

1. Starting from the well-known description of Leśniewski's symbols for 1- and 2-place $\frac{s}{s}$ and $\frac{s}{s\,s}$ functors of Protothetic we investigate the difficulty of keeping this style for $n \geq 3$ - place functors $\frac{s}{s_1 \ldots s_n}$. Some hints of such a method have been given by Luschei, but, in fact, no general effective method of defining such functors has been clearly and explicitly given. Our purpose is to provide such a method.

   Stanisław Leśniewski was the author of three systems: Protothetic, Ontology, and Mereology.[1] For the simplest of this systems, Protothetic, he has introduced quite a peculiar notation. This notation has the odd property that, in some sense, functors[2] are matrices of themselves, and some logical operations are equivalent to some geometrical modifications of functors. However, the standard notation provides only symbols for 1- and 2- place functors. We shall start with a standard description of this notation, and next, we shall proceed to develop a method for constructing symbols of functors of more than two arguments. Our presentation of 1- and 2- place functors will be developed in accordance with (Luschei, [31, p. 289 ff.]).[3]

2. Next, we are interested in the so-called Ontological Functors of Leśniewski's Elementary Ontology. By *Elementary Ontology* (LEO) we mean this part of Leśniewski's Ontology, in which the only variables are name-variables (and hence, quantifiers quantify only name-variables).

   A system is *functionally complete if and only if* all its possible functors may be defined with the use of 1-, and 2- place its functors as the only functors. By 'System S is *functionally complete with respect to the set of logical constants* $\{f_1, \ldots, f_k\}$.' we mean that all its possible functors may be defined with the use of $\{f_1, \ldots, f_k\}$ as the only logical constants. By 'System S is *F-functionally complete with respect to the set of logical*

[1] The first two systems are logical, the third one is an extralogical theory built on the former two. Extralogical, since it introduces an extralogical functor 'is a part of'.

[2] For convenience, we say 'functors' instead of 'sentence- forming functors of propositional argument(s)', when there is no danger of misinterpretation.

[3] With some modifications. Luschei's book is written in a quite difficult language which is a little bit foreign to the modern standard language of logic. We shall try to clarify the case.

constants $\{f_1, \ldots, f_k\}$' we mean that all the system's functors from the set of functors F may be defined with the use of elements of $\{f_1, \ldots, f_k\}$ as the only logical constants.

The purpose of this section is to provide an effective method of defining any ontological functor (This notion will be explicated later on. Tentatively - those are the functors specific to LEO.) of LEO with the use of $\{\wedge, \neg, \forall, \varepsilon\}$ only, and thus to sketch a proof for the thesis that LEO is functionally OF-complete with respect to $\{\wedge, \neg, \forall, \varepsilon\}$ (where OF denotes here the set of ontological functors).

3. After doing that, we propose quite an intuitive understanding of the expression: 'an axiom (or: an axiomatic basis) determines the meaning of the only specific constant occurring in it'. We introduce some basic semantics for functors of category $\frac{s}{n,n}$ of Leśniewski's Ontology. Using this results we prove that the popular claim that axiom(s) of Ontology (or axiomatic basis) determine(s) the meaning of primitive constant(s) is false.

   Intuitively (in a slogan), when we give axioms for a given axiomatic system, one of our purposes is to characterize constants occurring in these axioms. Following this idea, axiom(s) of Leśniewski's Ontology aim(s) to characterize 'univocally' primitive constant(s) of this system. Usually, there is only one such a constant specific to Ontology; it is ε (sometimes, other constants — see (Lejewski [24])). Hence, Lejewski writes:

   > In the original system of Ontology ... the meaning of the copula 'is' (' ε ' in symbols) is determined axiomatically... (Lejewski, [26, p. 323])

Our purpose will be to investigate, whether in fact axiomatizations of Ontology determine a unique semantic interpretation of the primitive constant(s) of this axiomatizations. In order to proceed, we shall (i) introduce the language we will be talking about (ii) say what axioms and rules of inference were accepted in Ontology in some axiomatizations, (iii) present some possible interpretations of quantifiers in Ontology, (iv) explain what is meant by 'semantic interpretation of a given functor', and, when it will be done, (v) obtain the answer for the main problem.

4. For understanding the importance of Leśniewski's systems (hence — also of his choice of functors used in their formulation), one cannot neglect his solution to Russell's paradox. Sobociński has given a formalization of both: Leśniewski's formulations of Russell's paradox and the Leśniewski's solution to it. Our purpose is to show that this solution (at least, as formalized by Sobociński) is to some extent insufficient, because the notion of distributive class used in this solution leads to other paradoxes, unnoticed by Sobociński.

5. The main purpose of our last section is three-fold: to elaborate an interpretation of categorical propositions that allegedly makes Syllogistic work as free logic, to decide which commonly accepted syllogistic rules hold both in non-empty and empty domains of objects named (both with the assumption of non-emptiness of common names, and without it), and to compare the rules thus obtained with some historical approaches to the Syllogistic of categorical propositions. The secondary purpose is to show that there are some passages in Aristotle's *Organon* as well as in some works of other logicians that the currently commonly accepted interpretation of Syllogistic fails to explain, but which can be understood on the ground of the considered interpretation. The third purpose is to define some groups of syllogistic languages and to give definitions of model, satisfaction, truth and validity for this languages.

Please note, that the symbolic notation varies a little bit in different sections, sice it is accustomed to the purposes of given section.

2 Functors of Protothetic

2.1 Application of Functors and Quantification

Expressions of Protothetic can be built from:

- Propositional variables $p, q, r, s, p_1, q_1, r_1, s_1, \ldots, p_n, q_n, r_n, s_n$.
- Sentence-forming functors of n sentence arguments[4] (to be introduced).
- The universal quantifier (In some presentations of Protothetic the particular quantifier is introduced — but then it is defined by means of the universal one).
- Any constant of any semantic[5] category, which is correctly defined by means of the formerly introduced symbols.
- Whenever a category has been introduced, variables can be introduced which belong to this category.

Functors are written before arguments. Arguments are placed in parentheses and are not separated from each other by means of commas.

Universal quantification over variables of any semantic category $\varphi_1, \ldots, \varphi_n$ is noted: $\lfloor \varphi_1, \ldots, \varphi_n \rfloor$. The scope ψ of a quantifier is distinguished in the following manner: $\lceil \psi \rceil$.

Only sentences (expressions without free variables) can be well-formed expressions of Protothetic.

2.2 1- place Functors

The basic outline of each 1-place functor is a horizontal line:'$-$'. By adding to this line vertical lines on its ends (two such lines are possible) in all possible combinations we obtain all possible 1-place functors. The rules for this operations are simple:

- A vertical line occurs on the left side, iff (if the argument has the value 0, the whole expression has the value 1).

[4]For convenience, we do not make any distinction between sentences and propositions.

[5]We shall consequently use the expression *semantic category* instead of *syntactic category*, since this is the original name introduced by Leśniewski.

- A vertical line occurs on the right side, iff (if the argument has the value 1, the whole expression has the value 1).

Hence, we obtain four possible 1-place functors: $\dashv, \vdash, \vdash\!\dashv, -$, which can be described by the following classical matrix:

p	$\vdash\!\dashv$	$-$	$\dashv$	$\vdash$
1	1	0	1	0
0	1	0	0	1

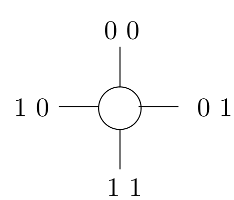
Functor $\vdash\!\dashv$ is called *verum* (Słupecki), or *Tautologous one argument connector* (Luschei). Functor $-$ is called *falsum* (Słupecki). [6] Functor $\vdash$ to is negation, and $\dashv$ – assertion.

2.3 2- place Functors

The basis of a 2-place functor is a circle: '$\circ$'. To this circle four possible lines can be added in four directions - up, down, left, right. The rules of adding this lines are as follows:

- A line to the left occurs iff for arguments having values respectively 1, 0, the whole expression has the value 1.
- A line to the right occurs iff for arguments having values respectively 0, 1, the whole expression has the value 1.
- A line upwards occurs iff for arguments of value 0, 0, the whole expression takes the value 1.
- A line downwards occurs iff for arguments of value 1, 1, whole expression takes value 1.

These rules can be pictured by the following schema:



[6]Luschei: *Contradictory connector of one argument.*

According to this rules we obtain 16 possible 2-place functors. To characterize them, we shall apply the method used by Church. (Church, [11, p. 31]). A 2-place functor will be characterized by a 4-place sequence of elements 1 and 0, simply by giving values taken by the whole expression when arguments have values, respectively, $11, 10, 01, 00$.[7]

Functor	Name	Characteristics
	Equivalence	1001
	Disjunction	0110
	Conjunction	1000
	Exclusion	0111
	Conegation	0001
	2-place *verum*	1111
	2-place *falsum*	0000
	Distinction	0100
	Contradistinction	0010
	Implication	1011
	Counterimplication	1101
	Alternation	1110
	Antecedent Affirmation	1100
	Consequent Affirmation	1010
	Antecedent Negation	0011
	Consequent Negation	0101

As we have said, the peculiar feature of this notation is that there is a correspondence between geometrical properties of symbols of functors and the logical properties of these functors. We shall not expand upon this subject. However, here are two examples:

1. Let us define the relation of inclusion between two functors:

$$\mathrm{D}\subset\!: \quad \lfloor fh \rfloor \lceil \; (\lfloor rs \rfloor \lceil \; (f(rs)h(rs)) \rceil \subset (fh)) \rceil$$

   As we can see, we have defined an expression of the category $\frac{s}{\frac{s}{ss}\,\frac{s}{ss}}$.

[7]Names of functors are usually given according to the Luschei's translation. There are some slight differences. Luschei calls equivalence *coimplication*, verum — *tautologous connection of two arguments*, falsum — *contradictory connection of two arguments*. Please note, that our nomenclature differs from the standard one (because if follows Leśniewski).

Inclusion of 1-place functors is defined analogically (i.e. we should change the number of arguments of functors and the number of variables which the quantifier binds).

Now we can say: a functor h of the same number of arguments as f includes functor f iff f is graphically a part (proper or not) of h.

For example, the functor: $\overset{|}{\underset{|}{\circ}}$ includes the functor $\overset{|}{\circ}$, because, for any r, s it is true, that

$$\text{if } \overset{|}{\circ}(r,s) \text{ then } \overset{|}{\underset{|}{\circ}}(r,s)$$

We can make use of the defined terminology by saying that two functors are equivalent iff they include each other.

2. We now shall define the functor *cnv* of the category $\frac{\frac{s}{ss}}{\frac{s}{ss}}$ by using a definition of a many-linked functor:

$$\mathrm{D}\, cnv\langle f\rangle : \qquad \lfloor frs \rfloor \lceil \overset{|}{\underset{|}{\circ}} (f(sr) cnv\langle f\rangle(rs)) \rceil$$

We may say that g is a converse of f iff g is equivalent to $cnv\langle f\rangle$. Now we can give a 'geometrical' rule:

Two 2-place functors, say f, g are converses of each other iff notationally:

(a) f has a horizontal line on a given side iff g has a horizontal line on the opposite side, and

(b) f has a vertical line in a given direction iff g also has a vertical line in the same direction.

For example the following pairs are converses of each other: $\overset{|}{\circ}, \overset{|}{\circ}$; $\overset{|}{\circ}\!-, -\!\overset{|}{\circ}$; $\circ\!-, -\!\circ$.[8]

[8]The same rule in Luschei's formulation is a little bit more complicated:

> Any binary connectors logically are converse, each having the force of a converse of the other, if and only if notationally they are converse, each having and indicator 180° opposite only any indicator of the other with respect to a y-axis (i.e. vertical through its hub in its own plane), so that the basic outline of each is equiform to that of the other converted by 180° rotation around such an axis.(Luschei, [31, p. 293])

2.4 $n \geq 3$ - place Functors

As it is claimed by Luschei (Luschei, [31, p. 299]):

> ...any connection of $n + 1$ arguments is logically equivalent to an alternative Boolean expansion developed according to the scheme:
>
> (A) $\quad \multimap\!\!\!\!\!\!\varphi\,(\,\varphi\,(F(p_1, \ldots, p_n)p_{n+1}) \multimap (G(p_1, \ldots, p_n)p_{n+1}))$

Next, he introduces a convention (not explicitly), that if for a given $n + 1$-place functor h there are functors f and g such that

(DEFINITION SCHEME)

$$\lfloor p_1, \ldots, p_{n+1} \rfloor \ulcorner \varphi(h(p_1, \ldots, p_{n+1}) \multimap\!\!\!\!\!\!\varphi\,(\varphi\,(f(p_1, \ldots, p_n)p_{n+1}) \multimap (g(p_1, \ldots, p_n)p_{n+1})))\urcorner$$

we can instead of h write fg.[9]

Luschei next gives some hints how to proceed to obtain symbols for functors. Nevertheless, his description has the following properties:

- It is not a clear general procedure and to large extent it requires intuition.
- There is no answer to the question how to find a symbol and a definition for a given functor with a given matrix.[10]
- It is complicated.[11]

On the other hand, we shall try to present a method which is a short and simple procedure. We give an explicit answer to the question: how to find a symbol and definition for a given functor with a given matrix.

2.5 Procedure

Consider the formula (A). It is an alternative. The first argument is a conjunction and the second - a distinction.[12] What is the difference between conjunction and distinction? The difference lies only in the value of the last argument.

[9] In the description of the notation of $n \geq 3$-place functors Luschei follows (Standley, [44]).

[10] Ideographic notation is introduced *without using special tables of truth-value analysis* (Luschei, [31, p. 303]).

[11] This is only an intuition. The reader is invited to check (Luschei, [31, pp. 289-305]) to estimate this claim.

[12] Please note, that in our text the expression 'distinction' is used to name a functor of Protothetic or an expression built from this functor and its arguments; It has very little to do with the normal use of this word.

The basic idea is that the first part of this alternative tells us for what values of first n arguments our functor h takes 1 as its value if the $n+1$-th argument has the value 1. The second part of this alternative tells us for what values of first n arguments our functor h takes 1 as its value if the $n+1$-th argument has the value 0.

What we mean by that, should be clear from the following. Let us consider a 3-place functor H as an example. There is no controversy that it can be characterized by a matrix. Let this functor have the following matrix:

p	q	r	H(pqr)
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	1

As we have said, we shall be interested in giving 'truth-conditions' of the expression $H(pqr)$, i.e. to indicate, for what values of arguments the whole expression has the value 1. Hence, we are interested only in the following part of the above matrix :

p	q	r	H(pqr)
1	1	1	1
1	1	0	1
1	0	1	1
0	0	0	1

Remark It has been suggested to us that it is not clear why exactly it should suffice (in order to characterize a given functor fully) to enumerate all valuations of arguments for which the whole expression has the value 1, without telling anything about '0-cases'. Hence we devote this remark to this subject.

Assume that we shall enumerate all the valuations of p, q, r in which $H(pqr)$ has the value 1. By doing so and saying that these are all such valuations, we implicitly give information about the valuations of p, q, r in which $H(pqr)$ has the value 0 — these are simply all valuations other than the explicitly mentioned.

We can make it clear by means of the example of ϕ with its well-known matrix. When we want to say that it has 1 as its value whenever either both arguments have 1 as values or both arguments have 0 as values, we say that this functor is definable by means of (has the same matrix as):

$$\text{Ex1}\qquad \text{-⚲-}(\text{⚲}\,(pq)\,\text{⚲}\,(\vdash(p)\vdash(q)))$$

We do not need to say: 'any other cases do not occur', because no two valuations of arguments can be 'considered simultaneously'. If we attribute a given valuation to arguments of a functor we cannot simultaneously attribute other values to them — it is a simple consequence of the fact that any valuation is a function. Hence, for equivalence, we do not have to add that

$$\text{Ex2}\qquad \text{⚲}\,(\vdash(\text{⚲}\,(p\vdash(q))))\vdash(\text{⚲}\,(\vdash(p)q))$$

because Ex2 is implied by Ex1. △

We have remarked, that in the alternative of the kind of (A) the first argument correspond to the cases when the last argument of h has value 1, and the second argument of this alternative corresponds to the situation when the last argument of h has the value 0.

Therefore, we shall separately consider two parts of the last table. The first:

p	q	r	H(pqr)
1	1	1	1
1	0	1	1

corresponds to the first argument of the alternative (A), and the second:

p	q	r	H(pqr)
1	1	0	1
0	0	0	1

to the second argument of this alternative.

Let us concentrate on the situation when the value of r is 1. We have to find such a 2-place functor f, that $\text{⚲}\,(f(pq)r)$ exactly when one of the mentioned two cases takes place. Since in these cases the value of r is 1, we are interested in a 2-place functor, which takes value 1 iff p and q has values either, respectively, 1, 1, or 1, 0 (according to this part of table, in which we are now interested). Obviously, according to the description of 2-place functors, it is -⚲. Therefore, the first part of our alternative has the form:

$$\text{⚲}\,(\text{-⚲}\,(pq)r)$$

We now turn to the second argument of the considered alternative, i.e. to this part of table, where $H(pqr)$ has the value 1, and r has the value 0. We are now looking for a 2-place functor g such that

$$\text{-○}\,(f(pq)r)$$

has the value 1 iff the variables p, q, r have the value either, respectively 1, 1, 0, or 0, 0, 0. Since r has the value 0 in both cases, and is the last argument of distinction, we now will be interested in finding a functor of two arguments, which yields the value 1 iff applied to arguments having the values either 1, 1, or 0, 0. Obviously it is the functor ⌽. Hence, our second argument of alternative will be:

$$\multimap(\text{⌽}(pq)r)$$

and the whole alternative (i.e. the equivalent of $H(pqr)$) will be:

$$\text{-⫯-}(\text{⫯}(\text{-⫯}(pq)r)\multimap(\text{⌽}(pq)r))$$

and our 3-place functor can be represented by:

$$\text{-⫯}\ \text{⌽}$$

Let us now consider the 3-place functor *falsum*, which for any arguments takes the value 0. There is no need to provide a matrix — it is obvious. How shall we construct a symbol for such a functor? Still, the main idea is that our alternative of the form (A) should yield the same value as our functor (i.e. *falsum*). In constructing such an alternative, we cannot proceed exactly as before. Before, we have first chosen these lines of the table, where the functor yielded the value 1. Now, we cannot do the same, since there are no such lines. We therefore need such f and g, that:

$$\text{-⫯-}(\text{⫯}(f(pq)r)\multimap(g(pq)r))$$

would take 0 as its value for any values of variables occurring in this formula. It is an alternative, therefore we need both:

(i) a functor f that would make ⫯$(f(pq)r)$ take 0 as value for any values of arguments, and

(ii) a functor g that would make $\multimap(g(pq)r))$ take 0 as its value for any values of arguments.

In both cases, it is the 2-place functor *falsum*: ∘ Therefore, the symbol for 3-place *falsum* is:

$$\circ\ \circ$$

We may give a general instruction for symbolizing and defining $n \geq 3$-place functors. Let us note that for a $n \geq 3$-place functors, the number of symbols of 2-place functors used in its symbol is equal to 2^{n-2}. For example, for three arguments it is $2^{3-2} = 2$, for four arguments it is $2^{4-2} = 4$, for five it is $2^{5-2} = 8$, etc.

INSTRUCTION 1 *1. Select from the matrix of h those lines in which there is 1 in the column of $h(p_1, \ldots, p_{n+1})$ (if there are any). Let these lines be called H1 lines of this matrix.*

2. *If there are no such lines, h is represented by a 2^{n-2}-place sequence of repeated '$\circ$' symbol. The definition of such a functor is obtained from the DEFINITION SCHEME by substituting two n-place falsum functors for f and g.*

3. *Divide the H1 lines in two groups: the first, where there is 1 in the column of p_{n+1} (let this group be called H1A), and the second, where there is 0 in the column of p_{n+1} (let this group be called H1B).*

4. *Consider the H1A group. List all valuations of $p_1, \ldots, p_n$ occurring in H1A. Obtain an n-place functor which yields 1 iff the arguments have one of these valuations. This is the functor f we were looking for.*

5. *Consider the H1B group. List all valuations of $p_1, \ldots, p_n$ occurring in H1B. Obtain an n-place functor which yields 1 iff the arguments have one of these valuations. This is the functor g we were looking for.*

6. *To obtain the notation, concatenate: fg. To obtain the definition, substitute such obtained functors for, respectively, f and g in the DEFINITION SCHEME.*

For example, we shall consider a 4-place functor H characterized by the following table :

p	q	r	s	H(pqrs)
1	1	1	1	1
1	1	0	1	1
1	0	1	1	1
1	0	0	1	0
0	1	1	1	0
0	1	0	1	0
0	0	1	1	0
0	0	0	1	1
1	1	1	0	0
1	1	0	0	0
1	0	1	0	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	0	0

The H1 group is:

1	1	1	1	1
1	1	0	1	1
1	0	1	1	1
0	0	0	1	1

As we can see, in this particular case, H1 is identical with H1A, and H1B is empty. We consider H1A. We are looking for the 3-place functor which yield 1 iff the arguments have one of the following valuations: 1, 1, 1, or, 1, 1, 0, or, 1, 0, 1, or 0, 0, 0. As we already now from our examples of 3-place functors, it is ⫟ ⫰. Now we consider $H1B$, which is empty. Since it is empty, we must find a 3-place functor which yields 1 for no valuation. Obviously, it is the already introduced $\circ\ \circ$. Hence, we can represent H by means of:

$$⫟⫰\circ\ \circ$$

and the definition of H is:

$$\lfloor p_1, \ldots, p_4 \rfloor \lceil ⫰(h(p_1, \ldots, p_4) ⫟ (⫯ (⫟⫰ (p_1, \ldots, p_3) p_3) ⊸ (\circ\circ\ (p_1, \ldots, p_3) p_4))) \rceil$$

2.6 Leśniewskian functors as strings of 1's and 0's

As a remark, we can add that this notation can be translated into the 0's and 1's talk. 1-place functors can be represented by 2-place strings of 1's and 0's. 2-place functors can be represented by 4-place strings of 0's and 1's. Say, each place of a string corresponds to a fixed direction in Leśniewski's notation; 1 occurs in this place iff the Leśniewskian symbol has a line in the direction corresponding to this place in the string. Concatenation of Leśniewskian functors corresponds to the concatenation of strings. Geometric operations correspond to easily definable operations composed of erasing and writing 0's and 1's in a string.

3 Ontological Functors and their Definability

3.1 Basic Notions

We omit definitions of $\exists$, $\vee$, $\rightarrow$, $\equiv$ by means of $\{\wedge, \neg, \forall\}$ as obvious.

Let $\{a, b, a_1, \ldots, a_n b_1, \ldots, b_n\}$ be a set of name variables of Ontology. Next, we apply the following convention for variables in meta-language: $\varphi, \psi, \varphi_1, \psi_1, \ldots, \varphi_n, \psi_n$ represent sentences; $\chi, \chi_1, \ldots, \chi_n$ represent sentential-expressions (including sentences and sentential formulas); $\tau, \tau_1, \ldots \tau_n$ represent sentential formulas; $\mu, \nu, \mu_1, \nu_1, \ldots, \mu_n, \nu_n$ represent name-variables; $\pi, \pi_1, \ldots \pi_n$ represent names; $\alpha, \alpha_1, \ldots, \alpha_n$ represent name-expressions (including names and name-formulas). Let $\sigma^1, \ldots, \sigma^n$ be variables[13] each of which can be substituted only by a name of a Semantic Status (the notion will be defined later on).[14] Let also $\delta, \delta^1, \ldots, \delta^n$ represent functors of category $\frac{s}{n_1,\ldots,n_k}$.[15] Let $L, L_1, \ldots, L_n$ represent languages. The variables $f, f_1, \ldots, f_n$ represent functions.

We distinguish: *Unshared* names, each of which names exactly one object (e.g. 'Socrates'). According to the long-lasting tradition, we accept the view that names signify without time. It means that if a name names an object which does not actually exist, but either existed or is going to exist, the name is not empty. However, this assumption is not an essential one. We could have assumed the opposite, without any loss of accuracy. *Shared* names, each of which names more than one object (e.g. 'egg'). *Fictitious* names, i.e. expressions which regards their syntax behave like shared or unshared names, but which do not name anything (e.g. 'Gandalf', 'Unicorn').[16]

3.1.1 *LEO*-Languages

We start with syntactical definitions of languages for which LEO-systems can be built. We shall start from the languages without variables, which are the simplest cases of LEO-languages. For convenience, we shall not distinguish

[13]In the case of this variables they are differentiated by upper case numbers, since the lower case number informs us about the number of names the Semantic Status of which we are talking.

[14]We also use the symbols of the same shape as logical constants of Ontology in meta-language for naming these constants.

[15]This variables are differentiated by upper case numbers, since the lower case number informs us about the number of arguments of these functors.

[16]The conduct of this distinction is due to (Lejewski, [24]).

between languages and its algebras of expressions. We introduce the notion of LEO-language without variables ($CLEO$): [17]

3.1.2 $CLEO$-Languages

Definition 1 $L \in CLEO \equiv L = \langle S^L_C, N^L_C, \varepsilon, \wedge, \neg \rangle$

Where N^L_C *is the set of name constants (it does not matter: shared, unshared, or fictitious)*[18]*,* ε *is a primitive functor of the category* $\frac{s}{n_1, n_2}$*,* $\neg, \wedge$ *are classical extensional functors of Sentential Calculus, here treated as primitive ones.* S^L_C *is the set of well built sentences, being the least set which fulfills the following conditions:*[19]

1. $\ulcorner \pi_1 \urcorner, \ulcorner \pi_2 \urcorner \in N^L_C \rightarrow \ulcorner \varepsilon\langle \pi_1, \pi_2 \rangle \urcorner \in S^L_C$
2. $\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner \in S^L_C \rightarrow \ulcorner \neg\varphi \urcorner, \ulcorner \varphi \wedge \psi \urcorner \in S^L_C$

We introduce the notion of a model for $CLEO$-languages.

Definition 2 $\mathcal{M}$ *is a model for* $L \in CLEO \equiv \mathcal{M} = OBJ$

where OBJ *is a set of objects.*

We introduce the *function of extension of names*, from N^L_C into 2^{OBJ}. The set of such a functions is EXT^{OBJ}_L. In our further deliberations we will often assume that the model and language are settled, and denote the extension of π simply as '$Ext(\pi)$'. We will also use variables representing extension functions of L in $\mathcal{M}$, denoted by $f^{\mathcal{M}}_L$, or f^{OBJ}_L.

Definition 3 $\mathcal{I}$ *is an interpretation of* $L \in CLEO \equiv \mathcal{I} = \langle \mathcal{M}, f^{\mathcal{M}}_L \rangle$*.*

The *Valuation function –* $Val^{\mathcal{I}}_L$ *– of a* $CLEO$ *language, say* L*, in an interpretation* $\mathcal{I}$ is a function that maps S^L_C onto[20] $\{0, 1\}$ in the way defined below (Let $\mathcal{I}$ be $\langle OBJ, f^{\mathcal{M}}_L \rangle$):

Definition 4 $Val^{\mathcal{I}}_L$ *is the function from* S^L_C *onto* $\{0, 1\}$*, satisfying the conditions:*

---

[17] 'C' in '$CLEO$' coming from 'Constants'.

[18] As the elements of N_C (and later on, of N_{CV}, to be defined) in a language we will use $\{c, d, c_1, d_1, \ldots, c_n, d_n\}$.

[19] We use quasi-quotation marks, i.e. the expression: $\ulcorner \varphi \urcorner$ is the name of the sentence ϕ.

[20] We could as well have said: 'into'. Since we have closed the set S^L_C under the operation of classical negation, it makes no difference, the result is the same.

1. $Val_L^{\mathcal{I}}(\ulcorner \varepsilon\langle \pi_1, \pi_2\rangle \urcorner) = 1 \equiv \exists! x \in f_L^{\mathcal{M}}(\pi_1) \wedge f_L^{\mathcal{M}}(\pi_1) \subset f_L^{\mathcal{M}}(\pi_2)$

2. $\ulcorner \varphi \urcorner \in S_C^L \rightarrow [Val_L^{\mathcal{I}}(\ulcorner \neg\varphi \urcorner) = 1 \equiv Val_L^{\mathcal{I}}(\ulcorner \varphi \urcorner) = 0]$

3. $\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner \in S_C^L \rightarrow [Val_L^{\mathcal{I}}(\ulcorner \varphi \wedge \psi \urcorner) = 1 \equiv Val_L^{\mathcal{I}}(\ulcorner \varphi \urcorner) = 1 \wedge Val_L^{\mathcal{I}}(\ulcorner \psi \urcorner) = 1]$

Now, obtaining the definition of *truth in a given interpretation* ($T_{\mathcal{I}}$) is trivial:

Definition 5 $T_{\mathcal{I}}(\ulcorner \varphi \urcorner) \equiv Val_L^{\mathcal{I}}(\ulcorner \varphi \urcorner) = 1$

3.1.3 $CAVELEO$-Languages

Now we extend our languages, allowing it to contain name constants and variables. We define the set of $CAVELEO$-languages.[21]

Definition 6 $L \in CAVELEO \equiv L = \langle FS_{CV}^L, NV_{CV}^L, \varepsilon, \wedge, \neg, \forall \rangle$

where NV_{CV}^L is the union of N_{CV} (which is the set of name constants of L) and V_{CV} (which is the set of name variables of L). [22] *The functor ε is a primitive functor of category $\frac{s}{n_1, n_2}$, $\neg$, $\wedge$ are classical extensional functors of Sentential Calculus, here treated as primitive ones, $\forall$ is simply the universal quantifier (it did not occur in CLEO-languages, since there were no variables to quantify over).*

FS_{CV}^L is the union of S_{CV}^L (which is the set of sentences of L), and F_{CV}^L (which is the set of propositional formulas of L). It means that FS_{CV}^L is the least set satisfying the following conditions:

1. $\ulcorner \alpha_1 \urcorner, \ulcorner \alpha_2 \urcorner \in NV_{CV}^L \rightarrow \ulcorner \varepsilon\langle \alpha_1, \alpha_2 \rangle \urcorner \in FS_{CV}^L$

2. $\ulcorner \chi_1 \urcorner, \ulcorner \chi_2 \urcorner \in FS_{CV}^L \rightarrow \ulcorner \neg \chi_1 \urcorner, \ulcorner \chi_1 \wedge \chi_2 \urcorner \in FS_{CV}^L$

3. $\ulcorner \mu \urcorner \in V_{CV}^L \wedge \ulcorner \chi \urcorner \in FS_{CV}^L \rightarrow \ulcorner \forall_\mu \chi \urcorner \in FS_{CV}^L$

The definition of model for $L \in CAVELEO$ is the same, as before (definition 2 on page 21). Similar situation occurs with respect to the extension function. It maps N_{CV}^L into 2^{OBJ}. The interpretation of a $CAVELEO$-language consists also in giving the model and extension function.

[21]'$CAVE$' in '$CAVELEO$' coming from 'Constants And VariablEs'.

[22]As elements of V_{CV} (and later on, of V_V, to be defined) in a language we will use $\{a, b, a_1, b_1, \ldots, a_n, b_n\}$.

Some difficulties arise, when we want to consider the valuation of variables, and the truth of expressions containing variables. For we can either emphasize that they are NAME variables, or that they are name VARIABLES. The question is: should we valuate a name variable *via* names, or not?

If we choose the first option, consequently, we can allow only these valuations which can be 'obtained' by means of substitution of name variables by names as well. Namely, we must agree that (the extension function $f^{\mathcal{M}}_L$ is given), when we understand a valuation of name variables V^L_{CV} as a sequence $\langle A_1, \ldots, A_n \rangle = A^u$ of n elements of 2^{OBJ}, we have to exclude from possible valuations such tuples A^u for which $\exists_{A_i} \neg \exists_{\pi \in N^L_{CV}} [A_i = f^{\mathcal{M}}_L(\pi)]$.[23] On the other hand, if we choose the second option, we put no restriction on A^u, but accordingly concede that there are such valuations of name variables for which there are no corresponding names. We could avoid this difficulty by the simple assumption that $\forall_{L \in CAVELEO} \forall_{A_i \in 2^{OBJ}} \exists_{\pi \in N^L_{CV}} [A_i = f^{\mathcal{M}}_L(\mu)]$. Unfortunately, languages which do not fulfill this condition seem to be quite legitimate objects of investigation.

For convenience, we have decided to define the valuation of variables for $CAVELEO$-languages in accordance to the first option, and to leave the most general concept of valuation for $VALEO$-languages (to be defined), which do not contain name constants. The interesting result is that some sentences built by preceding a sentential formula by a universal quantification can be true in a $CAVELEO$-language L just because of the nature of the set of names of L.[24]

To any name we can attribute a set that is its extension (i.e. denotation). In this way, every tautology (or valid expression) in the wider sense of valuation is a tautology (valid formula) in the narrower sense of valuation.

The question is, as we have said, whether the implication in the other direction is true. The answer would be simple, if we assumed the mentioned additional condition. Practically, it seems, however, that we are lacking names. As Ajdukiewicz argues (Ajdukiewicz, [1, p. 138]):

> Names of each language divide into simple and composed. There is always a finite number of the simple ones, the composite names

[23] We have loosely said, that a set A_i belongs to A^u; we meant that it is an element of the sequence, obviously.

[24] For instance, if we have $OBJ = \{1, 2, 3\}$, $N^L_{CV} = \{c, d\}$, $f^{\mathcal{M}}_L(c) = \{1, 2\}$, $f^{\mathcal{M}}_L(d) = \{3\}$, it is true in this interpretation that $\forall_{a,b}(\neg(\varepsilon\langle a, b\rangle \wedge\ \varepsilon\ \langle b, a\rangle) \rightarrow \neg\ \varepsilon\ \langle a, b\rangle)$.

are always finite combinations of simple names, hence there are $\aleph_0$ names.

If we simply take a universe containing the set of natural numbers (or any other universe of the power $\aleph_0$), according to Cantor's theorem, the number of subsets of the universe will be greater than $\aleph_0$.

Nevertheless, we can claim the following:

Theorem 1 *If language L fulfills the following requirement:*

> *For any formula of elementary theory of numbers, if this formula contains exactly one free variable, there is in L a general name, extension of which is identical with the set of numbers satisfying this formula.*[25]

then it is true, that any formula of L valid in the lexical sense, is valid in the semantic sense.

The full proof is to be found in (Pietruszczak, [35]).

We define the notion of valuation of name variables in $CAVELEO$-languages.

Definition 7 *The valuation of* $V^L_{CV} = \{\mu_1, \ldots, \mu_k\}$ *is a sequence:* $\langle A_1, \ldots, A_k\rangle = A^u$ *of elements of* 2^{OBJ} *such that* $\forall_{A_i}\exists_{\pi\in N^L_{CV}}[A_i = f^{\mathcal{M}}_L(\pi)]$.

The value of μ_i in an interpretation A^u will be denoted as $A^u(\mu_i)$, or simply A_i.

We define the notion of *Satisfaction*:

Definition 8 *We assume, that the sequences: of names and name variables are fixed.*

1. *Sentence* $\ulcorner\varepsilon\langle\pi_1,\pi_2\rangle\urcorner$ *is satisfied in* $\mathcal{I} = \langle\mathcal{M}, f^{\mathcal{M}}_L\rangle$ *by a valuation* A^u if and only if $\exists!_x x \in f^{\mathcal{M}}_L(\pi_1) \wedge f^{\mathcal{M}}_L(\pi_1) \subset f^{\mathcal{M}}_L(\pi_2)$. *It is obvious, that, since* $\ulcorner\varepsilon\langle\pi_1,\pi_2\rangle\urcorner$ *does not contain variables (or, in other words, '*A^u*' does not occur* in definiente*), if there is at least one* A^u *satisfying the given sentence in* $\mathcal{I}$*, this sentence is satisfied by any valuation in* $\mathcal{I}$*.*

[25] Elementary theory of numbers is what we can say about natural numbers in terms of addition, multiplication, identity, sentence connectors, variables representing natural numbers, and quantifiers binding them, without introducing the notion of set.

2. *Let us consider such an expression* $\lceil\varepsilon\langle\alpha_i,\alpha_j\rangle\rceil$*, in which there is at least one name variable. Obviously, there are three cases:*
   (a) $\lceil\varepsilon\langle\pi_i,\mu_k\rangle\rceil$ *is satisfied in* $\mathcal{I} = \langle\mathcal{M}, f_L^{\mathcal{M}}\rangle$ *by a valuation* A^u if and only if $\exists!_x x \in f_L^{\mathcal{M}}(\pi_i) \wedge f_L^{\mathcal{M}}(\pi_i) \subset A_k$.
   (b) $\lceil\varepsilon\langle\mu_k,\pi_i\rangle\rceil$ *is satisfied in* $\mathcal{I} = \langle\mathcal{M}, f_L^{\mathcal{M}}\rangle$ *by a valuation* A^u if and only if $\exists!_x x \in A_k \wedge A_k \subset f_L^{\mathcal{M}}(\pi_i)$.
   (c) $\lceil\varepsilon\langle\mu_k,\mu_i\rangle\rceil$ *is satisfied in* $\mathcal{I} = \langle\mathcal{M}, f_L^{\mathcal{M}}\rangle$ *by a valuation* A^u if and only if $\exists!_x x \in A_k \wedge A_k \subset A_i$.
3. $\lceil\neg\chi\rceil$ *is satisfied in* $\mathcal{I} = \langle\mathcal{M}, f_L^{\mathcal{M}}\rangle$ *by a valuation* A^u if and only if $\lceil\chi\rceil$ *is not satisfied in* $\mathcal{I} = \langle\mathcal{M}, f_L^{\mathcal{M}}\rangle$ *by a valuation* A^u.
4. $\lceil\chi_i\wedge_j\rceil$ *is satisfied in* $\mathcal{I} = \langle\mathcal{M}, f_L^{\mathcal{M}}\rangle$ *by a valuation* A^u if and only if $\lceil\chi_i\rceil$ *is satisfied in* $\mathcal{I} = \langle\mathcal{M}, f_L^{\mathcal{M}}\rangle$ *by a valuation* A^u *and* $\lceil\chi_j\rceil$ *is satisfied in* $\mathcal{I} = \langle\mathcal{M}, f_L^{\mathcal{M}}\rangle$ *by a valuation* A^u.
5. $\lceil\forall_{\mu_k}\chi\rceil$ *is satisfied in* $\mathcal{I} = \langle\mathcal{M}, f_L^{\mathcal{M}}\rangle$ *by a valuation* A^u if and only if $\lceil\chi\rceil$ *is satisfied in* $\mathcal{I} = \langle\mathcal{M}, f_L^{\mathcal{M}}\rangle$ *by any possible valuation* A^d *which differs from* A^u *at most on* $k-th$ *place.*

We define the notion of truth in an interpretation ($T_{\mathcal{I}}$):

Definition 9 $T_{\mathcal{I}}(\lceil\chi\rceil) \equiv \forall_u[\chi \textit{ is satisfied by } A^u \textit{ in } I]$

Given sentential expression is true in an interpretation, if it is satisfied by every possible in this interpretation valuation of (its) name variables.

Obviously, we can define validity as being true in any interpretation.

3.1.4 $VALEO$-Languages

We introduce languages without name constants, but containing name variables. We define the set of $VALEO$-languages.[26]

Definition 10 $L \in VALEO \equiv L = \langle FS_V^L, V_V^L, \varepsilon, \wedge, \neg, \forall\rangle$

whereas V_V^L *is the set of name variables of* L*,* ε *is a primitive functor of category* $\frac{s}{n_1,n_2}$*;* $\neg$*,* $\wedge$ *are classical extensional functors of Sentential Calculus, here treated as primitive ones,* $\forall$ *is simply the universal quantifier.*

[26]'VA' in '$VALEO$' coming from 'VAriables'.

FS^L_V is the union of S^L_V (which is the set of sentences of L [27]), and F^L_V (which is the set of propositional formulas of L). It means that FS^L_V is the least set satisfying the following conditions:

1. $\ulcorner\mu_1\urcorner, \ulcorner\mu_2\urcorner \in V^L_V \rightarrow \ulcorner\varepsilon\langle\mu_1, \mu_2\rangle\urcorner \in FS^L_{CV}$
2. $\ulcorner\chi_1\urcorner, \ulcorner\chi_2\urcorner \in FS^L_V \rightarrow \ulcorner\neg\chi_1\urcorner, \ulcorner\chi_1 \wedge \chi_2\urcorner \in FS^L_V$
3. $\ulcorner\mu\urcorner \in V^L_V \wedge \ulcorner\chi\urcorner \in FS^L_V \rightarrow \ulcorner\forall_\mu\chi\urcorner \in FS^L_V$

The definition of model is the same, as definition 2 on page 21:

Definition 11 *$\mathcal{M}$ is a model for $L \in VALEO \equiv \mathcal{M} = OBJ$ where OBJ is a set of objects.*

Since we use no names, we shall not need the notion of extension of a name. The interpretation of a $VALEO$-language consists only in giving a model. Therefore, we shall not define interpretation (model will suffice). We define the notion of valuation of name variables in $VALEO$-languages.

Definition 12 *The valuation of $V^L_V = \{\mu_1, \ldots, \mu_k\}$is a sequence:*

$$\langle A_1, \ldots, A_k\rangle = A^u \text{ of elements of } 2^{OBJ}$$

The value of μ_i in an interpretation A^u will be denoted as $A^u(\mu_i)$, or simply A_i.

We define the notion of *Satisfaction*:

Definition 13 *We assume, that the sequence of variables is fixed.*

1. *$\ulcorner\varepsilon\langle\mu_k, \mu_i\rangle\urcorner$ is satisfied in $\mathcal{M}$ by a valuation A^u* if and only if $\exists!_x x \in A_k \wedge A_k \subset A_i$.
2. *$\ulcorner\neg\chi\urcorner$ is satisfied in $\mathcal{M}$ by a valuation A^u* if and only if *$\ulcorner\chi\urcorner$ is not satisfied in $\mathcal{M}$ by a valuation A^u.*
3. *$\ulcorner\chi_i \wedge \chi_j\urcorner$ is satisfied in $\mathcal{M}$ by a valuation A^u* if and only if *$\ulcorner\chi_i\urcorner$ is satisfied in $\mathcal{M}$ by a valuation A^u and $\ulcorner\chi_j\urcorner$ is satisfied in $\mathcal{M}$ by a valuation A^u.*
4. *$\ulcorner\forall_{\mu_k}\chi\urcorner$ is satisfied in $\mathcal{M}$ by a valuation A^u* if and only if *$\ulcorner\chi\urcorner$ is satisfied in $\mathcal{M}$ by any valuation A^d which differs from A^u at most on $k-th$ place.*

[27] Nota bene, though there are no names, it is possible to obtain sentences from formulas by use of quantifier.

We define the notion of truth in model ($T_{\mathcal{M}}$):

Definition 14 $T_{\mathcal{M}}(\ulcorner\chi\urcorner) \equiv \forall_u[\chi$ *is satisfied in* $\mathcal{M}$ *by* $A^u]$

Given sentence expression is true in an interpretation, if it is satisfied by every possible in this model valuation of name variables. Validity is defined as truth in any model.

3.2 The so-called Nominalism of LEO-systems

Leśniewski, in fact, was a nominalist. Hence, sometimes, his systems (systems formulated in *LEO*-languages, *a fortiori*) are believed to be nominalistic. However, the deliberations hitherto led show us only that what suffices as a model of given *LEO*-language, is a set of objects.

This statement obliges use neither to accept, nor to refute nominalism. We have decided neither what kind of objects can belong to the set OBJ, nor what object must belong to this set.

Moreover, it is far from being clear, that belonging to a model is to be interpreted as real existence. Thus, even, if we had decided, that a model $\mathcal{M}$ of an *LEO*-language L can contain only objects of a particular sort, still it would not force us to accept any claim either about real existence of anything, or about the lack of it.

3.3 Notion of Semantic Status (SeS)

3.3.1 Intuitions

There are some specific functors which we will be considering in this section. Those are *Ontological Functors* (OF). They are those specific functors which distinguish *Elementary Ontology* from *Prototetics.* As it is obvious, these functors are of syntactical category $\frac{s}{n_1,\ldots,n_c}$.

However, this information does not suffice for distinguishing OF-s from other functors of the same syntactical categories. For example, we would consider the expression 'is' in: 'Socrates is mortal.' an OF. This functor is of the syntactical category $\frac{s}{n,n}$; but we can as well find a functor of the same syntactical category, which surely is not an OF, e.g. the expression 'loves' in 'John loves Mary.'.

Hence we need some other condition (or a set of conditions), which would not only be necessary, but also sufficient for a functor to be an OF. In what follows, we shall fulfill this requirement.

The state of affairs affirmed by propositions built from an OF and its name arguments is called the (possible) semantic status (SeS) of this arguments.

3.3.2 1-place SeS-es

Definition 15 *Semantic Status of a given name π is I.i. ($1 \geq i \geq 3$), whereas i is equal (respectively) to:*

$$\begin{cases} 1 & iff & \exists! x[x \in Ext(\pi)] \\ 2 & iff & \exists x[x \in Ext(\pi)] \wedge \neg\exists! x[x \in Ext(\pi)] \\ 3 & iff & \neg\exists x[x \in Ext(\pi)] \end{cases}$$

Instead of saying: 'an SeS which is k-place' we will use lower case numbers: 'SeS_k' is a name (shared) with $Ext(SeS_k)$ identical with the set of all k-place SeS-es.[28]

Example 1 The SeS_1 of the name *Socrates* is I.1.. △[29]

Instead of saying 'the SeS_k of k names (in given order) $\langle \pi_1, \ldots, \pi_k \rangle$' we will simply write: '$SeS_k\langle \pi_1, \ldots, \pi_k \rangle$'. We also, where there will be no danger of ambiguity, will use the notation of SeS_k as a function of k arguments $\langle \pi_1, \ldots, \pi_k \rangle$ with the Semantic Status of $\langle \pi_1, \ldots, \pi_k \rangle$ as value.

3.3.3 2-place SeS

Now we proceed to defining all possible 2-place SeS-es.

From now on, if we are talking about a semantic status of k names, it is to be assumed, that we consider the order of these names important. We sometimes write instead of 'a SeS of k names (in order)' simply 'SeS of k names', just for convenience.

[28]For clarity of presentation, we will sometimes use the abbreviations introduced for the terms defined, just as if they were nouns. We will also use their singular and plural forms. The singular forms are identical with abbreviations themselves; The plural ones are constructed by adding the endings: -s, or -es. Strictly speaking, our definitions of SeS-es are definitions of functions. However, when we use our symbols otherwise, it is made for the sake of presentation, and, we believe, there is no danger od ambiguity.

[29]We will use the symbol '△' as indicating the end of an example.

Definition 16 *Semantic Status of two names (in order)* $\langle \pi_1, \pi_2 \rangle$ *is* $II.i$ *(*$1 \geq i \geq 16$*), where* i *is equal (respectively) to:*

$$
\begin{cases}
1 \; iff \; SeS_1\langle \pi_1 \rangle = SeS_1\langle \pi_2 \rangle = I.1. \wedge Ext(\pi_1) = Ext(\pi_2) \\
2 \; iff \; SeS_1\langle \pi_1 \rangle = SeS_1\langle \pi_2 \rangle = I.1. \wedge Ext(\pi_1) \neq Ext(\pi_2) \\
3 \; iff \; SeS_1\langle \pi_1 \rangle = I.1. \wedge SeS_1\langle \pi_2 \rangle = I.2. \wedge Ext(\pi_1) \subset Ext(\pi_2) \\
4 \; iff \; SeS_2\langle \pi_2, \pi_1 \rangle = II.3. \\
5 \; iff \; SeS_1\langle \pi_1 \rangle = I.1. \wedge SeS_1\langle \pi_2 \rangle = I.2. \wedge Ext(\pi_1) \not\subset Ext(\pi_2) \\
6 \; iff \; SeS_2\langle \pi_2, \pi_1 \rangle = II.5. \\
7 \; iff \; SeS_1\langle \pi_1 \rangle = I.1. \wedge SeS_1\langle \pi_2 \rangle = I.3. \\
8 \; iff \; SeS_2\langle \pi_2, \pi_1 \rangle = II.7. \\
9 \; iff \; SeS_1\langle \pi_1 \rangle = SeS_1\langle \pi_2 \rangle = I.2. \wedge Ext(\pi_1) = Ext(\pi_2) \\
10 \; iff \; SeS_1\langle \pi_1 \rangle = SeS_1\langle \pi_2 \rangle = I.2. \wedge Ext(\pi_1) \subset Ext(\pi_2) \wedge \\
\qquad \wedge Ext(\pi_1) \neq Ext(\pi_2) \\
11 \; iff \; SeS_2\langle \pi_2, \pi_1 \rangle = II.10. \\
12 \; iff \; SeS_1\langle \pi_1 \rangle = SeS_1\langle \pi_2 \rangle = I.2. \wedge \exists x[x \in Ext(\pi_1) \wedge \\
\qquad \wedge x \notin Ext(\pi_2)] \wedge \exists x[x \in Ext(\pi_2) \wedge x \notin Ext(\pi_1)] \wedge \\
\qquad \wedge \exists x[x \in Ext(\pi_1) \wedge x \in Ext(\pi_2)] \\
13 \; iff \; SeS_1\langle \pi_1 \rangle = SeS_1\langle \pi_2 \rangle = I.2. \wedge \neg \exists x[x \in Ext(\pi_1) \wedge \\
\qquad \wedge x \in Ext(\pi_2)] \\
14 \; iff \; SeS_1\langle \pi_1 \rangle = I.2. \wedge \\
\qquad \wedge SeS_1\langle \pi_2 \rangle = I.3. \\
15 \; iff \; SeS_2\langle \pi_2, \pi_1 \rangle = II.14. \\
16 \; iff \; SeS_1\langle \pi_1 \rangle = SeS_1\langle \pi_2 \rangle = I.3.
\end{cases}
$$

3.3.4 Ontological Table

The $SeS_1 - es$ and SeS_2-es hitherto defined can, perhaps, be better grasped, if we apply the graphical method of representing them. The method itself was used in (Lejewski, [24, p. 128]). Lejewski however, has not defined the notion of *Semantic Status* and has finished his semantic considerations on presenting the Ontological Table, which, for us, is just a point of departure for further investigations. Nevertheless, we present the Table, just as it occurs in (Lejewski, [24]), for convenience of the reader. By a shaded circle we represent the only object named by an unshared name. By an unshaded circle we represent the many objects each of which is named by a shared name. No circle will be used in case of fictitious name.

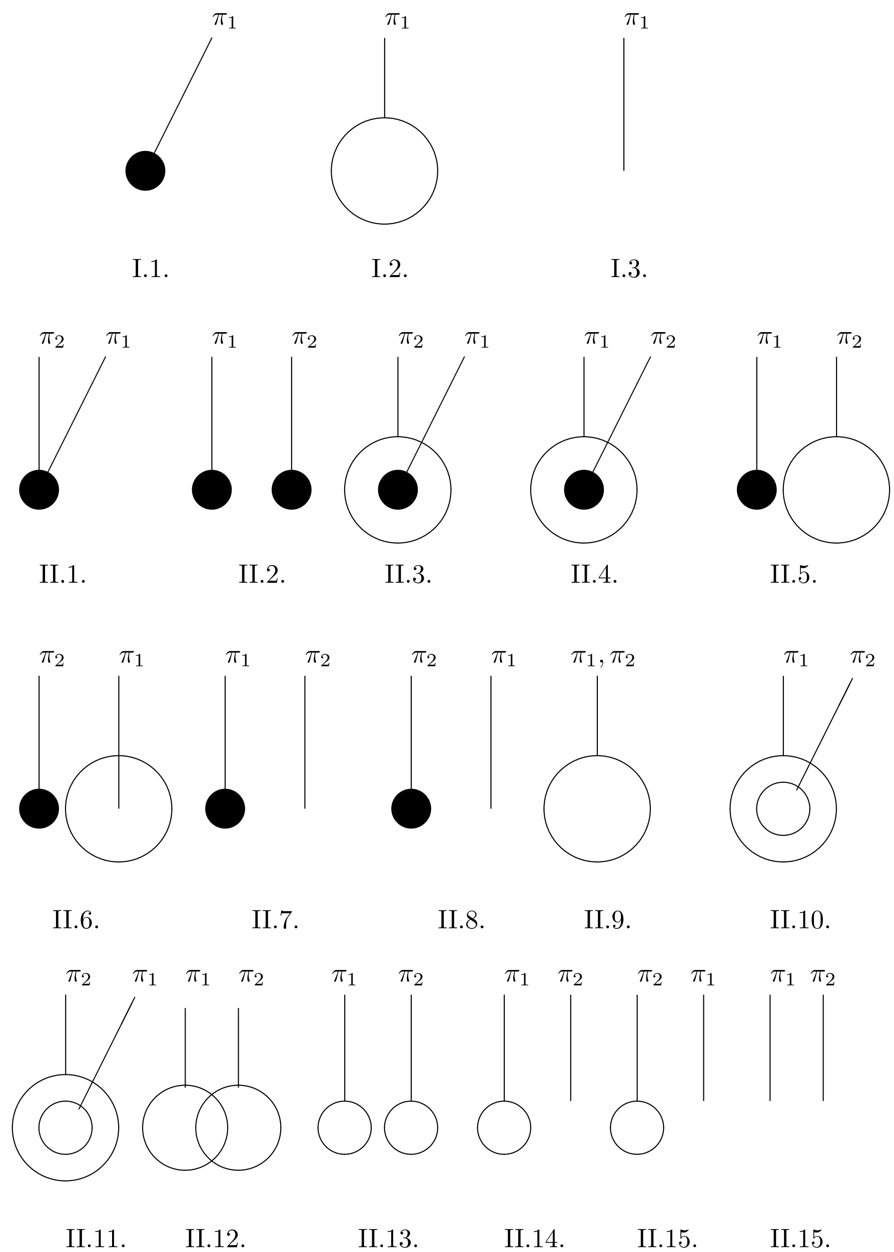


Figure 1: The Ontological Table

3.3.5 Identity, Union and Intersection of SeS-es

We define the relation of identity in the set of SeS-es. Two SeS-es, say σ_k^1, σ_m^2, can remain in this relation only if $k = m$:

Definition 17

$$\begin{aligned}&\sigma_k^1 = \sigma_m^2 \equiv m = k \wedge\\ &\wedge \forall_{\pi_1^1,\ldots,\pi_k^1}[SeS_k\langle \pi_1^1,\ldots,\pi_k^1\rangle =\\ &= \sigma_k^1\langle \pi_1^1,\ldots,\pi_k^1\rangle \equiv\\ &\equiv SeS_k\langle \pi_1^1,\ldots,\pi_k^1\rangle = \sigma_m^2\langle \pi_1^1,\ldots,\pi_k^1\rangle]\end{aligned}$$

Now we define the notion of *union of SeS-es*. Let us consider two sequences of names: $\pi_1^1,\ldots,\pi_k^1$ and $\pi_1^2,\ldots,\pi_m^2$. Let $\pi_1^{1,2},\ldots,\pi_c^{1,2}$ be all different name variables among $\pi_1^1,\ldots,\pi_k^1$ and $\pi_1^2,\ldots,\pi_m^2$.

Definition 18

$$\begin{aligned}&SeS_c\langle \pi_1^{1,2},\ldots,\pi_c^{1,2}\rangle = [\sigma_k^1\langle \pi_1^1,\ldots,\pi_k^1\rangle]\cup\\ &\cup[\sigma_m^2\langle \pi_1^2,\ldots,\pi_m^2\rangle] \equiv SeS_k\langle \pi_1^1,\ldots,\pi_k^1\rangle =\\ &= \sigma_k^1\langle \pi_1^1,\ldots,\pi_k^1\rangle \vee [SeS_m\langle \pi_1^2,\ldots,\pi_m^2\rangle =\\ &= \sigma_m^2\langle \pi_1^2,\ldots,\pi_m^2\rangle]\end{aligned}$$

We define the notion of *intersection of SeS-es*:

Definition 19

$$\begin{aligned}&SeS_{k+m}\langle \pi_1^{1,2},\ldots,\pi_c^{1,2}\rangle = [\sigma_k^1\langle \pi_1^1,\ldots,\pi_k^1\rangle]\cap\\ &\cap[\sigma_m^2\langle \pi_1^2,\ldots,\pi_m^2\rangle] \equiv SeS_k\langle \pi_1^1,\ldots,\pi_k^1\rangle =\\ &= \sigma_k^1\langle \pi_1^1,\ldots,\pi_k^1\rangle \wedge [SeS_m\langle \pi_1^2,\ldots,\pi_m^2\rangle =\\ &= \sigma_m^2\langle \pi_1^2,\ldots,\pi_m^2\rangle]\end{aligned}$$

3.3.6 ($n \geq 3$)-place SeS

We are going to introduce the notion of SeS of $n \geq 3$ names. We precede the formal definition by some intuitive deliberations. The SeS_1-es and SeS_2-es are states of affairs which concern names and pairs of names. Knowing the SeS_1 of a name, we know, whether there are no objects named by this name, or there is exactly one such an object, or there are more than one such objects. Knowing the SeS_2 of any given pair of names, we not only know, what the SeS_1 of each of these names is, but also, in what relation to each other remain the sets of their designates.

It is however quite natural, that the number of names which may be taken under consideration with respect to their SeS should not come to an end with

2. We can ask, in what 3-place semantic (to say it loosely) relation given three names remain.

Intuitively, we would allow all n-place (where $n \in \mathcal{N}$) relations between names to be n-place SeS-es, as far, as they would be representable by means of diagrams similar to these used in the Ontological Table in the Figure 1.

Quite helpful in defining $SeS_{n\geq 3}$, say σ^1, seems to be the fact, that if we have $\sigma^1_{n\geq 3}\langle \pi_1, \ldots, \pi_{n\geq 3}\rangle$, we know all SeS_2 of all pairs from $\{\pi_1, \ldots, \pi_{n\geq 3}\} \times \{\pi_1, \ldots, \pi_{n\geq 3}\}$, i.e. from $\{\pi_1, \ldots, \pi_{n\geq 3}\}^2$. Also, it seems to work in the other direction: if we know all SeS_2 of all pairs from $\{\pi_1, \ldots, \pi_{n\geq 3}\}^2$, we know what the $\sigma^1_{n\geq 3}\langle \pi_1, \ldots, \pi_{n\geq 3}\rangle$ is. Hence, (according to our intuitive deliberations and the definition 17 of identity) we define:

Definition 20
$$\sigma^1 = SeS_{k\geq 3}\langle \pi_1, \ldots, \pi_{k\geq 3}\rangle \equiv$$
$$\equiv \sigma^1 = \bigcap \left\{ \sigma^2_2\langle \pi_i, \pi_j\rangle : \langle \pi_i, \pi_j\rangle \in \{\pi_1, \ldots, \pi_{k\geq 3}\}^2 \right\}$$

In other words, a SeS is a SeS of more than two names *if and only if* it is identical with intersection of SeS_2-es of all pairs of names which were taken under consideration.

Definition 21 σ *is a* SeS if and only if *it is a* SeS_k*, for some* $k \in \mathcal{N}$

3.4 Theorems on SeS-es

Corollary 1 $\forall \pi[SeS_1(\pi) = I.1. \vee SeS_1(\pi) = I.2. \vee SeS_1(\pi) = I.3.]$

Corollary 1 claims that for every name π, there is among I.1., I.2., I.3. at least one Semantic State of π.

PROOF: Let B be a set-variable. It is true that:

$$\forall B[\exists! x[x \in B] \vee \exists x[x \in B] \wedge \neg\exists! x[x \in B] \vee \neg\exists x[x \in B]]$$

Since the Ext function takes sets as values, it is also the case that for all π:

$$\exists! x[x \in Ext(\pi)] \vee \exists x[x \in Ext(\pi)] \wedge \neg\exists! x[x \in Ext(\pi)] \vee \neg\exists x[x \in Ext(\pi)]$$

If $\exists! x[x \in Ext(\pi)]$, then, according to the definition given, $SeS_1(\pi) = I.1.$. If $\exists x[x \in Ext(\pi)] \wedge \neg\exists! x[x \in Ext(\pi)]$, and being so, according to the definition given, $SeS_1(\pi) = I.2.$. If $\neg\exists\ x[x \in Ext(\pi)]$, according to the definition given, then $SeS_1(\pi) = I.3.$. The formula: $(p \vee q \vee r) \rightarrow [(p \rightarrow s) \wedge (q \rightarrow t) \wedge (r \rightarrow u) \rightarrow (s \vee t \vee u)]$ is a tautology of Sentential Calculus. By a proper substitution, *modo ponendo ponente* we obtain the demanded corollary. Q.E.D.

In the similar manner, using set theory and the introduced semantics, we can easily prove the following few claims:

Corollary 2 $\forall\pi\neg\exists_{i\neq j}[SeS_1\langle\pi\rangle = I.i. \wedge SeS_1\langle\pi\rangle = I.j.]$

Lemma 1 $\forall\pi\exists!_i[SeS_1\langle\pi\rangle = I.i.]$

Corollary 3 $\forall\pi_1, \pi_2\exists_{1\leq i\leq 16}[SeS_2\langle\pi_1, \pi_2\rangle = II.i.]$

Corollary 4
$$\forall\pi_1, \pi_2\forall_{1\leq i,j\leq 16}[SeS_2\langle\pi_1, \pi_2\rangle = II.i.\wedge$$
$$\wedge SeS_2\langle\pi_1, \pi_2\rangle = II.j. \rightarrow i = j]$$

Lemma 2 $\forall\pi_1, \pi_2\exists!_{1\leq i\leq 16}[SeS_2\langle\pi_1, \pi_2\rangle = II.i.]$

Corollary 2 tells us that for every name π there is among I.1., I.2., I.3. at most one Semantic Status of π. Lemma 1 informs that each name has exactly one one-place Semantic Status. Corollary 3 says that for each two names (in order, obviously) their SeS_2 is identical with at least one of the SeS_2-es already defined. Corollary 4 claims that for every pair of names there is at most one SeS_2 such that it is the SeS of these names. Lemma 2 says, that for every two names there is exactly one SeS_2 among I.1. - 1.16. which is the SeS_2 of these names.

3.5 Ontological Functors (OF)

3.5.1 1-Place Ontological Functors ($OF_1 - s$)

Definition 22 *Functor δ_1 is a SOF_1* if and only if

$$\forall\pi\exists!_{1\leq i\leq 3}[\delta_1\langle\pi\rangle \equiv SeS_1\langle\pi\rangle = I.i.]$$

By writing '$i \neq j \neq k$' etc. we mean that i, j, k are distinct from each other.

Definition 23 *Functor* δ_1 *is a* POF_1 if and only if

$$\exists_{1\leq i\neq j\neq\ldots\leq 3}[\delta_1\langle\pi\rangle \equiv \underbrace{SeS_1\langle\pi\rangle = I.i. \vee \ldots \vee SeS_1\langle\pi\rangle = I.j.}_{\textit{the number of disjuncts is } 2 \textit{ or } 3}]$$

There is also one specific 2-place functor, neither SOF_2, nor POF_2, namely $Falsum$ - F_2. With its name arguments it gives a false sentence.[30]

Definition 24 *A functor* δ *is an* OF_1 if and only if

it is either a SOF_1*, or a* POF_1*, or* F_1

3.5.2 2-Place Ontological Functors ($OF_2 - s$)

Definition 25 *Functor* δ_2 *is a* SOF_2 if and only if

$$\forall_{\pi_1,\pi_2}\exists!_{1\leq i\leq 16}[\delta_1\langle\pi_1,\pi_2\rangle \equiv SeS_2\langle\pi_1,\pi_2\rangle = II.i.]$$

Definition 26 *Functor* δ_2 *is a* POF_2 if and only if

$$\exists_{1\leq i\neq j\neq\ldots\leq 16}[\delta_1\langle\pi_1,\pi_2\rangle \equiv SeS_2\langle\pi_1,\pi_2\rangle = II.i.\vee$$
$$\vee \ldots \vee SeS_2\langle\pi_1,\pi_2\rangle = II.j.]$$

There is also one specific 2-place functor, neither SOF_2, nor POF_2, namely $Falsum$ - F_2. With its name arguments it gives a false sentence.

Definition 27 *Functor* δ_2 *is an* OF_2 if and only if δ_2 *is either a* SOF_2*, or a* POF_2*, or* F_2*.*

3.5.3 $k \geq 3$-place Ontological Functors

Definition 28 *Functor* $\delta_{k\geq 3}$ *is a* $SOF_{k\geq 3}$ if and only if

$$\forall_{\pi_1,\ldots,\pi_k}\exists!_{\sigma_{k\geq 3}}[SeS_{k\geq 3}\langle\pi_1,\ldots,\pi_k\rangle = \sigma_{k\geq 3} \equiv \delta_{k\geq 3}\langle\pi_1,\ldots,\pi_k\rangle]$$

Definition 29 $\delta_{k\geq 3}$ *is a* $POF_{k\geq 3}$ if and only if

$$\exists_{\sigma_k^1\neq\ldots\neq\sigma_k^u}\forall_{\pi_1,\ldots,\pi_k}[\delta_{k\geq 3}\langle\pi_1,\ldots,\pi_k\rangle \equiv$$
$$\equiv \sigma_k^1 = SeS_k\langle\pi_1,\ldots,\pi_k\rangle \vee \ldots \vee \sigma_k^u = SeS_k\langle\pi_1,\ldots,\pi_k\rangle]$$

Definition 30 *Functor* $\delta_{k\geq 3}$ *is* $F_{k\geq 3}$ *(i.e.* $falsum$ *functor)* if and only if *with its k name arguments, it always gives a false sentence.*

[30] Clearly, it does not fit into definitions of SOF-s or POF-s.

Definition 31 *Functor* $\delta_{k\geq 3}$ *is an* $OF_{k\geq 3}$ if and only if *it is either a* $SOF_{k\geq 3}$*, or a* $POF_{k\geq 3}$*, or* $F_{k\geq 3}$*.*

Obviously, a given functor δ_k is an OF *if and only if* it is either an OF_1, or an OF_2, or an $OF_{k\geq 3}$.

Two k-place OF-s, say δ^1_k, δ^2_k are identical *if and only if* the truth conditions of sentences obtained by them and their arguments are the same:

Definition 32 $\delta^1_k = \delta^2_k \equiv \forall_{\pi_1,\ldots,\pi_k}[\delta^1_k\langle\pi_1,\ldots,\pi_k\rangle \equiv \delta^2_k\langle\pi_1,\ldots,\pi_k\rangle]$

3.6 Defining Ontological Functors

Lemma 3 $\overline{\overline{SeS_k}} = \overline{\overline{SOF_k}}$

PROOF: There is a function from the set SOF_k onto the set SeS_k. What remains to be shown, is that this function is $1-1$. While defining, we are distinguishing subsets of the set of all functors of $\frac{s}{n_1,\ldots,n_k}$. Now, for any given SeS_k, say σ_k, among all possible k-place functors of category $\frac{s}{n_1,\ldots,n_k}$, there is at least one functor, say δ^1, satisfying the conditions given in the definition of SOF_k, simply because the conditions given are not contradictory. Now, we show that it is unique. Do notice, that if there was any functor with the same truth-conditions, according to the definition 32 on p. 35, it would be identical with our δ^1. Next, if it had different truth conditions, it either would not be a SoF_K, or it would be a SoF_K corresponding to an another SeS_1. Hence, for every SeS_k there is exactly one SOF_k corresponding to it. Q.E.D.

In our metalanguage we will use variables already introduced. However, definitions which we will give according to the procedures (these procedures will be described below), will be introduced in an exemplary system of $VALEO$ language, in which name variables are $a, b, a_1,\ b_1, \ldots, a_n, b_n$. The definitions of OF-s from now on called: 'OF-Definitions' will be equivalences (following the style of Leśniewski[31]). For our purpose, the form of definition makes no difference.

[31]With the difference, that our definitions will not be preceded by universal quantifier(s). Of course, such an addition, for our purpose would not be essential.

3.6.1 OF_1-s

As it is discernible, there are exactly three $SOF_1 - s$. In order to systematize them, and to give some hints which allow to understand intuitively the method of defining OF-s further developed, let us construct following table:

$SOF_1 - s$ TABLE			
I.1.	I.2.	I.3.	Functor
1	0	0	*ob*
0	1	0	s (from 'shared')
0	0	1	fi (from 'fictitious')

To each of first three columns of this table there is a corresponding SeS_1. Each line of this table corresponds to an SOF_1. We put '1' in a column of a SeS_1, say σ, in the line of a SOF_1, say δ, to denote that the occurrence of σ of given name, say π, is a sufficient condition of the truth of $\delta\langle\pi\rangle$. We put '0' in the column of δ in the column of σ to denote that the non-occurrence of σ of π is a necessary condition of the truth of $\delta\langle\pi\rangle$.

Hence, we may introduce a convenient general method of referring to k-place functors:

Instruction 1 *After construction of an $m+1$-column table, where m is the number of possible different SeS_k , and each column $i-th$ from the left of this table corresponds to the $i-th$ of $SeS_k - es$ (their order to be fixed), and the column $m+1$ is left for placing functor-symbols, every OF_k, say δ, may be represented by a sequence consisting of m elements , each of which is 0 or 1. The $i-th$ element of the sequence is 1* if and only if *in the $i-th$ column, in the line of δ there is 1. Otherwise, it is 0. Obviously, there are exactly 2^m such sequences.*

Thus, the table just given may be extended to include all possible three place sequences of elements of $\{0,1\}$:[32]

[32]Where the names of functors have been already introduced in the history of Ontology, we simply use them. Where there are no such names, we introduce them.

$OF_1 - s$ TABLE			
I.1.	I.2.	I.3.	Functor
1	0	0	*ob*
0	1	0	*s* (from 'shared')
0	0	1	*fi* (from 'fictitious')
0	0	0	F_1 (from 'falsum')
1	1	0	*ex*
1	0	1	*sol*
1	1	1	V_1 (from 'verum')
0	1	1	*nob* (from 'non-object')

Now we define SOF_1 with the use of $\{\wedge, \neg, \forall, \varepsilon\}$ only.

OF-Definition 1 $ob\langle a\rangle \equiv \exists b[\varepsilon\langle a, b\rangle]$ [33]

OF-Definition 2 $s\langle a\rangle \equiv \exists b[\varepsilon\langle b, a\rangle] \wedge \neg\exists b[\varepsilon\langle a, b\rangle]$

OF-Definition 3 $f\langle a\rangle \equiv \forall b, c[\varepsilon\langle b, a\rangle \wedge \varepsilon\langle c, a\rangle \rightarrow$

$$\rightarrow \varepsilon\langle b, c\rangle \wedge \varepsilon\langle c, b\rangle] \wedge \neg\exists b\ [\varepsilon\langle b, a\rangle]$$

Thus we obviously have:

Lemma 4 *All SOF_1-s are definable by means of $\{\wedge, \neg, \forall, \varepsilon\}$ as the only logical constants.*

Here is the method of constructing a definition of any OF_1 which is not a SOF_1 by means of SOF_1-s:

Instruction 2 *1. Represent the functor to be defined $\delta_1\langle a\rangle$ by a* 3*-place sequence according to the INSTRUCTION 1 on page 36.*

2. For every place of the sequence here is exactly one formula corresponding to it; namely: '$ob\langle a\rangle$' to the first, '$s\langle a\rangle$' to the second, and '$fi(a)$' to the third.

3. Construct the conjunction of negations of all these three formulas if and only if *no element of the sequence is 1.*

[33]It is important, that (especially particular) quantifiers in our meta-language are interpreted differently from quantifiers in LEO. The first are understood existentially, the second are not.

4. If and only if *more than one element of the sequence is 1, construct the disjunction of formulas corresponding to these elements.(The case when exactly one element of the sequence is excluded, since this procedure is a procedure of defining OF_1-s which are not $SOF_1 - s$.)*

5. *As a result of those steps, a formula is obtained. Let it be τ. The formula τ is the right side of the definition. The left side is $\delta_1\langle a\rangle$. Construct the formula $\delta_1\langle a\rangle \equiv \tau$. this is the definition of δ_1.*

To make clear the proper understanding of this procedure, we will lead it for one functor:

Example 2 *Let us consider the functor 'ex'. It is represented by $\langle 1, 1, 0\rangle$ More than one element of this sequence is* 1*. Hence, we go to the step 4 and obtain $ob\langle a\rangle \vee s\langle a\rangle$. We obtain the definition:*

OF-Definition 4 $ex\langle a\rangle \equiv ob\langle a\rangle \vee s\langle a\rangle$ △

According to the procedure just characterized, we can proceed with the remaining definitions of OF_1-s. Therefore:

Lemma 5 *All OF_1-s are definable with the use of 'ε' as the only OF.*

PROOF: According to lemma 4 on p. 37 'ε' suffices as the only OF for defining all SOF_1-s. As we have seen, all OF_1 not being SOF_1-s are definable by means of SOF_1-s.[34] All SOF_1 have been defined by means of 'ε'. Therefore, we can any given expression (also any definition) containing any OF_1 other than replace by an equivalent formula with 'ε' as the only OF. We can do it also in definitions of OF_1-s, in which we have used SOF_1 *in definientibus*, thus obtaining *definitiones in quarum definientibus* 'ε' is the only OF. Q.E.D.

3.6.2 OF_2-s

We consider SOF_2-s. As it may be seen from the Ontological Table, there are 16 exactly different SeS_2. Therefore, we can obtain the number of possible OF_2-s. It is equal to the number of SOF_2-s.

[34]Since the usage of non-ontological functors in definitions is obvious, and we are mainly concerned with ontological functors, we omit the phrase: 'as the only OF' where it is obvious from the context.

For convenience, functors for which symbols have not been hitherto introduced in those part of the history of Leśniewski's Ontology which is known to the author of this text, will be symbolized by those sequences of elements of $\{0, 1\}$, which correspond to those functors similarly to the convention introduced for referring to OF_1-s. However, this convention requires an extension. First, we construct a similar table. The understanding of the last column of the table, will be explained in a moment.

$SOF_2 - s$ TABLE																
1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14	15	16.	OF
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	=
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	{2}
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	{3}
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	{4}
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	{5}
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	{6}
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	{7}
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	{8}
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	{9}
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	{10}
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	{11}
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	{12}
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	{13}
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	{14}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	{15}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	{16}

Instruction 3 *It will be convenient to represent OF_2-s, excluding F_2 and V_2 (which will be simply denoted by symbols 'F_2' and 'V_2'), in some other way than by* 16*-place sequences (as the instruction 1 on page 36 would suggest). First we construct the table being an extension of the SOF_2-s TABLE to OF_2-s TABLE. The extension consists in adding all possible sequences of* 0, 1 *as lines of the table. An OF_2, say δ_2, is represented by a set $\{k_1, \ldots, k_m\}$, $16 \geq k_1, k_m \geq 1$, where m is the number of occurrences of symbol '1' in the line of the OF_2-TABLE, which corresponds to this δ_2, and are numbers of columns in which '1' occurs, counting from left, and $\forall_{i,j}[k_i \neq k_j]$.*

Clearly, all SOF_2-s are definable by means of ε as the only OF.[35]

[35] I cannot prove it in general, so I simply show it, giving the required definitions below. To grasp the sense of those definitions it suffices to consider definitions hitherto given and The Ontological Table.

OF-Definition 5 $= \langle a, b\rangle \equiv \varepsilon\langle a, b\rangle \wedge \varepsilon\langle b, a\rangle$

OF-Definition 6 $\{2\}\,\langle a, b\rangle \equiv ob\langle a\rangle \wedge ob\langle b\rangle \wedge \neg\varepsilon\langle a, b\rangle$

OF-Definition 7 $\{3\}\,\langle a, b\rangle \equiv ob\langle a\rangle \wedge s\langle b\rangle \wedge \varepsilon\langle a, b\rangle$

OF-Definition 8 $\{4\}\,\langle a, b\rangle \equiv s\langle a\rangle \wedge ob\langle b\rangle \wedge \varepsilon\langle b, a\rangle$

Example 3 *The sentence formed from $\{2\}$ (corresponding to the second line of SOF_2-s TABLE) and its name arguments π_1, π_2 has exactly one sufficient truth-condition among SeS-es. Namely, the occurrence of the semantic status II.2. of $\langle \pi_1, \pi_2 \rangle$. Therefore we write '$\{2\}$'. The number of elements of this sequence 'tells us' that there is only one SeS_2, occurrence of which is the sufficient condition of the true of the sentence under consideration. The number '2' inside the brackets tells us which of all SeS_2-es it is.* △

Lemma 6 *All SOF_2-s are definable by means of 'ε' as the only OF.*

All POF_2-s can be exhaustively listed out by listing all possible 16-place sequences of elements of $\{0, 1\}$ such that more than one element of sequence is 1. According to instruction 3 on p. 39, it can be represented by a set of a specific kind (already described). We can now use our notation for introducing a general scheme od defining POF_2-s:

OF-Definition 9 $\{5\}\langle a, b\rangle \equiv ob\langle a\rangle \wedge s\langle b\rangle \wedge \neg\varepsilon\langle a, b\rangle$

OF-Definition 10 $\{6\}\langle a, b\rangle \equiv s\langle a\rangle \wedge ob\langle b\rangle \wedge \neg\varepsilon\langle b, a\rangle$

OF-Definition 11 $\{7\}\langle a, b\rangle \equiv ob\langle a\rangle \wedge fi\langle b\rangle$

OF-Definition 12 $\{8\}\langle a, b\rangle \equiv fi\langle a\rangle \wedge ob\langle b\rangle$

OF-Definition 13 $\{9\}\langle a, b\rangle \equiv s\langle a\rangle \wedge s\langle b\rangle \wedge \forall_c[\varepsilon\langle c, a\rangle \equiv \varepsilon\langle c, b\rangle]$

OF-Definition 14 $\{10\}\langle a, b\rangle \equiv s\langle a\rangle \wedge s\langle b\rangle \wedge \forall_c[\varepsilon\langle c, a\rangle \rightarrow \varepsilon\langle c, b\rangle] \wedge \neg\forall_c[\varepsilon\langle c, b\rangle \equiv \varepsilon\langle c, a\rangle]$

OF-Definition 15 $\{11\}\langle a, b\rangle \equiv s\langle a\rangle \wedge s\langle b\rangle \wedge \neg\forall_c[\varepsilon\langle c, a\rangle \rightarrow \varepsilon\langle c, b\rangle] \wedge \forall_c[\varepsilon\langle c, b\rangle \rightarrow \varepsilon\langle c, a\rangle]$

OF-Definition 16 $\{12\}\langle a, b\rangle \equiv s\langle a\rangle \wedge s\langle b\rangle \wedge \exists_{c_1}[\varepsilon\langle c_1, a\rangle \wedge \neg\varepsilon\langle c_1, b\rangle] \wedge$
$\wedge \exists_{c_2}[\varepsilon\langle c_2, a\rangle \wedge \varepsilon\langle c, b\rangle] \wedge \exists_{c_3}[\varepsilon\langle c_3, b\rangle \wedge \neg\varepsilon\langle c, a\rangle]$

OF-Definition 17 $\{13\}\langle a, b\rangle \equiv s\langle a\rangle \wedge s\langle b\rangle \wedge \neg\exists_c[\varepsilon\langle c, a\rangle \wedge \varepsilon\langle c, b\rangle]$

OF-Definition 18 $\{14\}\langle a, b\rangle \equiv s\langle a\rangle \wedge fi\langle b\rangle$

OF-Definition 19 $\{15\}\langle a, b\rangle \equiv fi\langle a\rangle \wedge s\langle b\rangle$

OF-Definition 20 $\{16\}\langle a, b\rangle \equiv fi\langle a\rangle \wedge fi\langle b\rangle$

We define the *falsum* 2-place functor F_2:

OF-Definition 21 $F_2\langle a, b\rangle \equiv ob\langle a\rangle \wedge \neg ob\langle b\rangle$

It is so called, because with its two name arguments it always yields a false proposition.

Instruction 4 *Follow the steps:*

1. *Represent the functor to be defined, say* δ_2*, by a set according to instruction 3.*

2. *For every number* $16 \geq i \geq 1$ *which may occur in the set such obtained, there is a formula corresponding to it. It is the formula built from* SOF_2*: '*$\{i\}$*' and its arguments* $\langle ab \rangle$*.*

3. *Any* POF_2 *is represented by a set* $\{k_1, \ldots, k_m\}$*, where* $16 \geq m \geq 2$*. Construct the disjunction of formulas corresponding to numbers occurring in this set.*

4. *As a result of those steps, a formula is obtained. Let it be* τ*. This* τ *is the right side of the definition. The left side is* $\delta_2\langle a, b\rangle$*. Construct the equivalence :* $\delta_2\langle a, b\rangle \equiv \tau$ *. This will be the definition looked after.*

Example 4 *We proceed to obtain a definition of the functor of strong inclusion which were listed among 14 OF-s in (Lejewski, [24, p. 129-130]). This functor usually it is noted by '* $<$*'. However, to avoid ambiguity of notation, we shall use '*$\prec$*'. This functor occurs in expressions of the type* $\prec \langle \alpha_1, \alpha_2 \rangle$*. A sentence built from '*$\prec$*' and its two name arguments is true* if and only if *exactly one of* SeS_2*-es occurs: II.1, II.3., II.9., II.10. Therefore, we can represent this functor by: '*$\{1, 3, 9, 10\}$*'.Consequently, we define it as follows (according to instruction 3):*

OF-Definition 22 $\prec \langle a, b\rangle \equiv = \langle a, b\rangle \vee \{3\} \langle a, b\rangle \vee \{9\} \langle a, b\rangle \vee \{10\} \langle a, b\rangle$ △

Hence:

Lemma 7 *All* OF_2*-s are definable by means of '*ε*' as the only OF.*

Proof: The set OF_2 is the union of SOF_2, POF_2, and $\{F_2\}$. All SOF_2-s are definable by means of 'ε' as the only OF (lemma 6). Functor 'F_2' is definable by means of 'ε' as the only OF. All POF_2-s are definable by means of SOF_2 as the only OF-s (instruction 4). By the transitivity of definability, we obtain the demanded result. q.e.d.

3.6.3 $OF_{k\geq 3}$-s

As we have said (definition 20) any $m \geq 3$-place SeS_m, say σ_m, of m different names, say $\langle \pi_1, \ldots, \pi_m \rangle$, is identical with the intersection of all SeS_2-es of all pairs being elements of $\{\pi_1, \ldots, \pi_m\} \times \{\pi_1, \ldots, \pi_m\}$. The deliberations hitherto led, suggest us that it is possible to define each m-place OF ($m \geq 3$) by means of OF_k-s ($2 \geq k$) as the only OF-s. How to execute such an operation?

If the number of SeS_m-es is settled, let it be k, the number of SOF_m is settled (since it is equal to the number of SeS_m-es -lemma 3 on p. 35) - it is k. If the number of SOF_m-s is settled, it is easy to determine the number of OF_m-s. It is the number of possible $\{0, 1\}$ variations of k-place sequence: 2^k. Hence:

Corollary 5 $\overline{\overline{OF_{m\geq 3}}} = 2^{\overline{\overline{SeS_{m\geq 3}}}}$

We could describe any $SeS_{m\geq 3}$ by describing all SeS_2-es of all ordered pairs being elements of $\{\pi_1, \ldots, \pi_m\} \times \{\pi_1, \ldots, \pi_m\}$. However, such a description would also contain some redundant information.

If we know the SeS_2 of π_1, π_2, we do not have to add any information regarding π_1, π_1, or π_2, π_2.The SeS_2 of π_i, π_i would inform us which of the states: II.1., II.9., or II.16. takes place between this name and itself. It is always some kind of identity. II.1., II.9 and II.16 differ only as to the question, whether I.1., I.2., or I.3. occurs. But such an information we have already got since we know the SeS_2 of π_1, π_2 , which determines not only the relation between π_1, π_2, but also SeS_1-es of π_1 and π_2. (do compare this statement with e.g. The Ontological Table). Next, if we know the SeS_2 of π_1, π_2, any additional information about the SeS_2 of π_2, π_1 is redundant.

Generally, for an m-place ($m \geq 3$) sequence of names $\langle \pi_1, \ldots, \pi_m \rangle$to determine their SeS_m is equivalent to determining the SeS_2-es of pairs: $\langle \pi_1, \pi_2 \rangle$, $\langle \pi_1, \pi_3 \rangle, \ldots, \langle \pi_1, \pi_m \rangle, \langle \pi_2, \pi_3 \rangle, \ldots, \langle \pi_2, \pi_m \rangle, \ldots, \langle \pi_{m-1}, \pi_m \rangle$.

There are $\frac{m^2-m}{2}$ such pairs. For each pair there are 16 possible SeS_2 that may take place for this pair. Therefore:

Corollary 6 *There are* $16^{\frac{m^2-m}{2}}$ *possible* $SeS_{m\geq 3}$*-es.*

Corollary 7 *There are* $2^{16^{\frac{m^2-m}{2}}}$ *possible* $OF_{m\geq 3}$*-s.*

Instruction 5 *Let $m \geq 3$. We want to construct an expression univocally denoting an m-place SOF_m, say δ_m.*

1. *If δ_m is a SOF_m, than there is only one SeS_m of the sequence $\langle \pi_1, \dots, \pi_m \rangle$, (let it be σ_m), such that $\delta_m \langle \pi_1, \dots, \pi_m \rangle$ is true* if and only if *σ_m of $\langle \pi_1, \dots, \pi_m \rangle$ takes place.*
2. *We reduce σ_m of $\langle \pi_1, \dots, \pi_m \rangle$ to the intersection of SeS_2-es: $\langle \sigma_2^1, \dots, \sigma_2^{\frac{m^2-m}{2}} \rangle$ of (respectively) $\langle \pi_1, \pi_2 \rangle$, $\langle \pi_1, \pi_3 \rangle$, ..., $\langle \pi_1, \pi_m \rangle$, $\langle \pi_2, \pi_3 \rangle$, ..., $\langle \pi_2, \pi_m \rangle$, ..., $\langle \pi_{m-1}, \pi_m \rangle$. Therefore we obtain the following sequence: $\langle \sigma_2^1, \dots, \sigma_2^{\frac{m^2-m}{2}} \rangle$*
3. *Each σ_2^i is one of 16 SeS_2-es. We construct k-place sequence $\langle s_1, \dots, s_k \rangle$ such that:*
   (a) $16 \geq s_i \geq 1$
   (b) $s_i = j \equiv \sigma_2^i = II.j.$
4. *To every number $16 \geq s_i \geq 1$ which may occur in the sequence such obtained, there is a formula corresponding to it. It is the formula built from SOF_2: '$\{s_i\}$' and its arguments $\langle a_x, b_y \rangle$, where x, y are the same indices which occur under the names in the pair corresponding to s_i.*
5. *Obtain the conjunction of all formulas $\tau_1, \dots, \tau_k$ corresponding to elements of $\langle s_1, \dots, s_k \rangle$. tel this conjunction be τ.*
6. *Construct the formula: $\delta_m \langle a_1, \dots, a_m \rangle$.*
7. *Construct the equivalence: $\delta_m \langle a_1, \dots, a_m \rangle \equiv \tau$. This is the definition looked after.*

Hence, we have:

Lemma 8 *All $SOF_{m \geq 3}$-s are definable by means of 'ε' as the only OF.*

PROOF: We can define them by means of SOF_2-s, which are themselves definable by means of 'ε'. Q.E.D.

Now, we proceed to the last phase of our deliberations. It remains to give the procedure of representing and defining all $OF_{m \geq 3}$-s not being $SOF_{m \geq 3}$-s. The definitions of $falsum$ functors are trivial, so we omit them. What remains to be given, is a general instruction for referring to $POF_{m\geq 3}$-s, and an instruction for defining them.

Instruction 6 *In order to represent any $POF_{m\geq 3}$, say $\delta_{m\geq 3}$, we have to:*

1. *Settle associated formulas (i.e. formulas being the result of applying a functor to its arguments, e.g. for δ_m it is $\delta_m\langle a_m, \ldots, a_m\rangle$) of all SOF_m -s in a sequence $\langle\tau_1, \tau_2, \ldots, \tau_k\rangle$ (Let k be the number of possible SOF_m-s). The order is non-essential (as far as we keep to the order once fixed).*

2. *When this order is fixed, for each POF_m δ_m there is exactly one sequence $\langle s_1, \ldots, s_k\rangle$ by which this POF_m may be represented, where each s_i ($k \geq i \geq 1$) is either 0, or 1, and $s_i = 1$* if and only if *τ_i having value 1 is sufficient condition of $\delta_m\langle a_1, \ldots, a_m\rangle$ having value 1; otherwise it is 0. We can represent therefore any δ_m by such kind of a sequence.*

3. *It is possible to list exhaustively all POF_m-s by writing out all possible variations of k-place sequence of elements belonging to $\{0, 1\}$.*

4. *Such a notation may be farther abbreviated for practical purposes. Namely, instead of writing $\langle s_1, \ldots, s_k\rangle$, we can write $\langle s'_1, \ldots, s'_c\rangle_m$, where for every i, $k \geq s'_i \geq 1$, c is the number of all elements of $\langle s_1, \ldots, s_k\rangle$ being equal to 1, and for any s_u, u occurs in $\langle s'_1, \ldots, s'_c\rangle_m$* if and only if *$s_u = 1$. The number m is equal to the number of arguments of the functor represented. However such a notation makes an exhaustive listing of possible functors and the description of defining procedure more complicated. Therefore, for purely theoretical purposes we still will use sequences of the kind: $\langle s_1, \ldots, s_k\rangle$.*

5. *This notation still can be abbreviated. The problem is, that if we stay on the level of what has been said in this instruction, all verum functors will remain unabbreviated, and all functors represented by sequences in which almost all elements are 1 will not lose much of the longitude of the sequences representing them. That is why we can, for practical purposes, complicate our instruction. Namely, instead of writing $\langle s_1, \ldots, s_k\rangle$, we can write:*

(a) *If the number of elements of* $\langle s_1, \ldots, s_k \rangle$ *being equal to 1 is not bigger than* $\frac{k}{2}$*:*
$\langle s'_1, \ldots, s'_c \rangle^1_m$, *where* c *is the number of all elements of* $\langle s_1, \ldots, s_k \rangle$ *being equal to* 1, *and for any* s_u, u *occurs in* $\langle s'_1, \ldots, s'_c \rangle$ if and only if $s_u = 1$. *The number* m *is equal to the number of arguments of the functor represented.*

(b) *If the number of elements of* $\langle s_1, \ldots, s_k \rangle$ *being equal to 1 is bigger than* $\frac{k}{2}$*:*
$\langle s'_1, \ldots, s'_c \rangle^0_m$, *where* c *is the number of all elements of* $\langle s'_1, \ldots, s'_c \rangle$ *being equal to 0, and for any* s_u , u *occurs in* $\langle s'_1, \ldots, s'_c \rangle$ if and only if $s_u = 0$. *The number* m *is equal to the number of arguments of the functor represented.*

We give a schema for defining POF_m-s:

Instruction 7 *Assuming that we want to define an* POF_m*, let it be* δ_m*.*

1. *We have settled the sequence of associated formulas (i.e. formulas being the result of applying a functor to its arguments, e.g. for* δ_m *it is* $\delta_m \langle a_m, \ldots, a_m \rangle$ *) of all* SOF_m *-s in a sequence* $\langle \tau_1, \tau_2, \ldots, \tau_k \rangle$ *(Let* k *be the number of possible* SOF_m*-s). We represented our* δ_m *by a sequence* $\langle s_1, \ldots, s_k \rangle$*, whereas each* s_i $(k \geq i \geq 1)$ *is either* 0*, or* 1*, and* $s_i = 1$ if and only if τ_i *having value 1 is sufficient condition of* $\delta_m \langle a_1, \ldots, a_m \rangle$ *having value 1; otherwise it is 0.*

2. *To any* s_i *there is a corresponding formula, namely* τ_i*. Form a disjunction of all formulas corresponding to those elements of the sequence* $\langle s_1, \ldots, s_k \rangle$*, which are equal to 1. Let the formula obtained be* τ*.*

3. *Form an associated formula of* δ_m *i.e.* $\delta_m \langle a_1, \ldots, a_m \rangle$*. Obtain the equivalence:* $\delta_m \langle a_1, \ldots, a_m \rangle \equiv \tau$*. This is the definition looked after.*

Lemma 9 *All* $OF_{m \geq 3}$*-s are definable by means of '*ε*' as the only ontological functor.*

Proof: All $SOF_{m \geq 3}$-s are definable by means of 'ε'as the only OF. All $POF_{m \geq 3}$-s are definable by means of $SOF_{m \geq 3}$-s, according to the procedure given above. Therefore, by extensionality for equivalence, and the fact, that definitions are equivalences, all $POF_{m\geq 3}$-s are definable by means of 'ε'as the only OF. [36] Q.E.D.

Next, we have:

Theorem 2 *All OF-s are definable by means of 'ε'as the only OF.*

PROOF: All OF_1-s, all OF_2-s, and all $OF_{m\geq 3}$-s are definable by means of 'ε'as the only OF. Therefore, all OF-s are definable by means of 'ε'as the only OF. Q.E.D.

[36] Obviously, we might have as well treated definitions as formulated in meta-language; the consequence would still hold, by definitional extensionality.

4 Non-Standard Interpretations

4.1 Basic Language (BL)

In our deliberations we will use a basic language which is somewhat simpler than the full language of Ontology. It will nevertheless allow us to give a short statement of the essential results of our section. We will suggest a possible way of extending our results to a richer language of Ontology. The presentation of our language is a simplified version of (Urbaniak, [47]).

Let $\{a, b, c, a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n\}$ be a set of name variables of Ontology, say V. Next, we apply the following convention for variables in meta-language: $\chi, \chi_1, \ldots, \chi_n$ represent sentential-expressions (including sentences and sentential formulae); $\mu, \nu, \mu_1, \nu_1, \ldots, \mu_n, \nu_n$ represent name-variables.[37] Let also $f, f_1, \ldots, f_n, g, g_1, \ldots, g_n$ represent functors of category $\frac{s}{n,\ldots,n}$. Sometimes we use in the meta-language other variables — their usage should be self-explanatory.

We introduce the following language. Our language L^f is relativized to a primitive functor f.

Definition 33 $L^f = \langle FS^f, V, f, \wedge, \neg, \forall \rangle$

where V is the already mentioned set of name variables of L^f, f is a primitive functor of category $\frac{s}{n_1,n_2}$; $\neg$, $\wedge$ are classical extensional functors of Sentential Calculus, here treated as primitive ones, $\forall$ is simply the general quantifier. We omit definitions of $\exists$, $\vee$, $\rightarrow$, $\equiv$ by means of $\wedge, \neg, \forall$, considering them quite obvious.

FS^f is the union of S^f which is the set of sentences[38] of L^f, and F^f, which is the set of propositional formulae of L^f. It means that FS^f is the least set satisfying the following conditions:

1. $\mu_1, \mu_2 \in V \rightarrow \ulcorner \mu_1 f \mu_2 \urcorner \in FS^f$
2. $\ulcorner \chi_1 \urcorner, \ulcorner \chi_2 \urcorner \in FS^f \rightarrow \ulcorner \neg\chi_1 \urcorner, \ulcorner \chi_1 \wedge \chi_2 \urcorner \in FS^f$
3. $\ulcorner \mu \urcorner \in V \wedge \ulcorner \chi \urcorner \in FS^f \rightarrow \ulcorner \forall_\mu \chi \urcorner \in FS^f$

[37] We also use the symbols of the same shape as logical constants in meta-language for naming these constants.

[38] For convenience, we do not distinguish between sentences and propositions.

For the sake of simplicity BL does not contain: variables of other category than n, constants different from name constants (introduced by means of the rule of definition) and functors of categories: $\frac{s}{n,n}$, $\frac{s}{s}$, $\frac{s}{s,s}$.

4.2 Interpretation of Quantifiers

The problem of interpretation of quantifiers in Ontology was discussed in the technical literature (e.g. (Luschei, [31, p. 109-112]), (Marcus [33]), (Kielkopf [19]), (Küng [21])). There is no unique solution all logicians would agree upon (though mainly the disagreement concerns technicalities).

The reason for such a controversy is the fact that the particular quantifier cannot be interpreted existentially (the meaning of the universal quantifier is also debatable, due to this fact), since we can prove in Ontology:

$$\exists_a \neg ex(a)$$

where 'ex' is to be read: 'exists' (Küng, [21, p. 315]).

Therefore different ideas of interpreting quantifiers have been suggested. We list some of them below:[39]

1. META-LANGUAGE INTERPRETATIONS – Ruth Barcan Marcus ([33, p. 252–253]) suggests that '$\exists_x Fx$' is to be read 'Some substitution instance of 'Fx' is true'. Following this idea, Küng distinguishes between substitutional and Leśniewskian reading of quantifiers:

   (a) SUBSTITUTIONAL READING - $\exists_a \phi(a)$ is to be read:

   > If some α of the same category as a is taken to be substituted for the inscription equiform to a, the following is asserted: $\phi(\alpha)$.

   (b) LEŚNIEWSKIAN READING - $\exists_a \phi(a)$ is to be read:

   > For some extension which the inscriptions equiform with a is taken to have, the following is asserted: $\phi(\alpha)$.

2. REFERENTIAL/OBJECTUAL INTERPRETATIONS — Kielkopf (Kielkopf, [19]) distinguishes some other interpretations of quantifiers which do not involve meta-language.

[39] While discussing the interpretation of quantifiers I could not use two important papers on this issue: (Rickey, [39]) and (Simons [40]). I regret that these papers were unavailable to me in the time of writing this text.

(a) REFERENTIAL-OBJECTUAL INTERPETATION — It consists in giving a set of objects, and claiming that quantifiers refer to them, in the sense, that $\exists_a \phi(a)$ is to be read:

For some object α of the given domain, $\phi(\alpha)$

(b) REFERENTIAL-NON OBJECTUAL INTERPRETATION — We choose a domain D and associate with D a non-empty set R called *realm*. The elements of R are not viewed as existing objects. They are viewed as mind-dependent entities and not 'fully existing objects'. Then, a language is given an referential-non objectual interpretation iff it is given a quasi-referential interpretation, as if R was a domain, but the set of existing objects is D.

3. MODEL-THEORETIC INTERPRETATION (Urbaniak [47]) — It consists in indicating a domain of D, considered to be the set of existing objects. The function of valuation on name-variables takes subsets of D as values. If a name is considered empty, then its denotation is Φ — the empty set.

Let us elaborate a while on the given possibilities of interpreting quantifiers. We shall not inquire, how far a given interpretation is Leśniewskian. Instead, we shall give some arguments that for our purpose it suffices to accept the model-theoretic interpretation.

First, let us consider the substitutional one. It is not quite clear, what names we can substitute for name variables. There are no 'just simply names'. Names always belong to some language. Therefore, such an interpretation of quantifiers either forces us to relativize our understanding of quantifiers to a given language, or requires that our quantifying involves implicit quantifying over languages (for example, we could read a universal quantification over a name variable as 'for any name in any language ...'). Moreover, it is far from being obvious that it is a really general understanding of quantifiers. What do we mean? That we can be lacking names for extensions.

To any meaningful name we can attribute a set that is its extension. In this way, every tautology (or valid expression) in the model-theoretic sense is a tautology (valid formula) in the substitutional sense. The question is, whether the implication in the other direction is true. The answer would be simple, if we assumed the condition that for any possible extension there is a name. Practically, it seems, however, that we are lacking names.[40]

From what has been already said, it follows that if we accept the model–theoretic interpretation, there is no loss of accuracy (in comparison with the substitutional interpretation).

Similar situation occurs in the case of so-called Leśniewskian interpretation. Since each possible extension of a meaningful name is a set, the model-theoretical interpretation does a good job. Moreover, it is not quite clear what the essential difference between this interpretation and the model-theoretic one (in terms of truth conditions) is.[41] Küng himself seems to agree that his interpretation implicitly contains the model-theoretical one:

> ... general procedure for reading all the functors of Leśniewski's 'ontology' in a way which is specifically Leśniewskian, ... which has at the same time the merit that it 'implicitly' contains the set-theoretical interpretation of those functors. (Küng, [21, p. 315])

Nevertheless, Küng's interpretation may have some virtues that the model-theoretical one does not have:

> In my opinion the question of how to read quantified statements is of some consequence. The habit of giving merely model—theoretic interpretations and no intuitive paraphrases has tended to obscure some subtle, but very important aspects of oblique speech. (Küng, [21, p. 309])

In our considerations we are not interested in aspects of oblique speech and we need a simple and user-friendly semantics. Fortunately, model theory formulated in the frame of standard set theory comes to the rescue.[42]

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[40] We have already discussed this issue.

[41] To some degree it may depend on whether we accept that there can be extensions of no actual names. This decision makes Leśniewskian interpretation either equivalent to the substitutional one, or to the model-theoretic one.

[42] Of course, one can claim that we cannot use standard set theory in meta-theory of Leśniewski's systems, because Leśniewski did not believe in distributive sets.

I will try to show on an example that *mutatis mutandis* the same results can be obtain within the Leśniewskian philosophical framework. However, I will not proceed according to this method. In our considerations, it would only complicate our reasonings.

Referential–objectual interpretation does not work, as far as we would like the valuation function to be from name-variables into the domain of object considered as existing individuals. First, it would lead to existential interpretation of particular quantifier. Next — it is not sure, how to interpret name-variables as semantically connected with exactly one existing object.

Nevertheless, the main idea of this interpretation is explicitly and successfully developed in (Takeuti, [45]). This semantics assumes that elements of a domain are sets (including the empty set which 'corresponds' to empty names). It surely does the job Takeuti claims it does, but we can equivalently use the model-theoretic semantics (developed below) which more emphasizes the fundamental role of individuals.

In the referential-non objectual interpretation the set of existing objects is D anyway. The valuation of name-variables takes subsets of R as values. Nevertheless, the value of propositional expressions does not depend on whether a value of a given name-variable is a non-empty subset of $R \setminus D$, or the empty set. Therefore, as far as we are interested in the truth of an expressions, we may as well accept the model-theoretical interpretation.

4.3 Standard Semantics

We now introduce the standard semantics for BL of Ontology, assuming that the primitive functor is ε .[43]

A model for L^f is a set OBJ, being a set of objects (interpreted as existing). Let the variables of L^f be $\mu_1, \ldots, \mu_n$, we assume that their order is fixed. A valuation of this variables is a sequence $A^u = \langle A_1, \ldots, A_n \rangle$, where, for every i, A_i is a subset of OBJ. Obviously, the value of μ_i in interpretation A^u is A_i. Sometimes, instead of A_i as the value of μ_i we shall write $V^u(\mu_i)$. For any name constant α its value (a subset of OBJ) is the same for any valuation of name variables.[44] We shall refer to this value in a given OBJ by '$V(\alpha)$', where it is clear which OBJ we mean. When it is not important whether α is a name constant or a variable, we refer to its semantic value in a given OBJ in a given valuation u of name variables by $EV^u(\alpha)$.

[43] We might have chosen otherwise. However, it is not an essential decision, as far as all other functors are definable by means of the primitive basis. The semantics for all other functors is to be constructed according to the introduced definitions of these functors.

[44] This account will be modified while discussing the rule of definition in non-standard interpretations.

We define the standard notion of satisfaction: satisfaction $^{\varepsilon}$. It is important that the quantifiers of our meta-language are interpreted differently from the quantifiers of our L^f. We interpret meta-language individual variables referentially (respective to OBJ).[45]

Definition 34 *We assume that the sequence of variables is fixed.*

1. $\lceil \mu_k \, \varepsilon \, \mu_i \rceil$ *is satisfied*$^{\varepsilon}$ *in* OBJ *by a valuation* A^u if and only if $\exists!_x x \in A_k \wedge A_k \subseteq A_i$.

2. $\lceil \neg\chi \rceil$ *is satisfied*$^{\varepsilon}$ *in* OBJ *by the valuation* A^u if and only if $\lceil \chi \rceil$ *is not satisfied*$^{\varepsilon}$ *in* OBJ *by the valuation* A^u.

3. $\lceil \chi_i \wedge \chi_j \rceil$ *is satisfied*$^{\varepsilon}$ *in* OBJ *by the valuation* A^u if and only if $\lceil \chi_i \rceil$ *is satisfied*$^{\varepsilon}$ *in* OBJ *by the valuation* A^u *and* $\lceil \chi_j \rceil$ *is satisfied*$^{\varepsilon}$ *in* OBJ *by the valuation* A^u.

4. $\lceil \forall_{\mu_k} \chi \rceil$ *is satisfied*$^{\varepsilon}$ *in* OBJ *by the valuation* A^u if and only if $\lceil \chi \rceil$ *is satisfied*$^{\varepsilon}$ *in* OBJ *by any possible valuation* A^d *which is different from* A^u *at most on one place, namely* $k-th$ *(place).*

A propositional expression is a tautology $^{\varepsilon}$ iff it is satisfied $^{\varepsilon}$ in any domain by any valuation.[46] In general, if we use upper indices for a given notion of satisfaction, say satisfaction$^{\psi}$, we say that validity$^{\psi}$ consists in being satisfied$^{\psi}$ in any domain by any valuation.

From now on, instead of saying 'an expression φ is satisfied$^{\psi}$ in OBJ by the valuation A$^{\delta}$' we say: 'φ is $SAT^{\psi,\delta}_{OBJ}$'.

4.4 Axioms

From among axioms of Ontology that were given in the development of Ontology we can distinguish axioms in which the only primitive specific functor of the system is ε , and axioms which either introduce also other functors beside ε , or introduce other functors instead of ε . Let us call the first ' ε –axioms', and the others '$\subset$–axioms'.

[45] We use the quantifier $\exists!$ which is to be read: 'there is exactly one'. Of course, it can be defined in terms of the universal quantifier and identity.

[46] In fact, Leśniewski excluded propositional formulae from the set of theorems of his system, so any tautology should be preceded by universal quantifiers binding the otherwise free variables. For our purpose the matter is inessential.

Historically, Ontology was usually based on a single axiom, however, there are at least a few possibilities of such an axiomatization. Let us list some main known axiomatizations.

4.4.1 ε -axioms

Historically, the following axioms (each one, as a single axiom of Ontology) were suggested (Lejewski, [24, p. 135–136]):

Axiom 1 $\forall_{a,b}[a \,\varepsilon\, b \equiv (\exists_c c \,\varepsilon\, a \wedge \forall_{c,d}(c \,\varepsilon\, a \wedge d \,\varepsilon\, a \rightarrow c \,\varepsilon\, d) \wedge \forall_c(c \,\varepsilon\, a \rightarrow c \,\varepsilon\, b))]$

The axiom 1 was firstly introduced by Leśniewski in 1920. In 1921 he proposed another axiom:

Axiom 2 $\forall_{a,b}[a \,\varepsilon\, b \equiv (\exists_c(c \,\varepsilon\, a \wedge c \,\varepsilon\, b) \wedge \forall_{c,d}(c \,\varepsilon\, a \wedge d \,\varepsilon\, a \rightarrow c \,\varepsilon\, d))]$

In the same year Sobociński proved that equivalently we can accept as an axiom:

Axiom 3 $\forall_{a,b}[a \,\varepsilon\, b \equiv (\exists_c(c \,\varepsilon\, a \wedge c \,\varepsilon\, b) \wedge \forall_c(c \,\varepsilon\, a \rightarrow a \,\varepsilon\, c))]$

Next simplification was introduced by Leśniewski in 1929:

Axiom 4 $\forall_{a,b}(a \,\varepsilon\, b \equiv \exists_c(a \,\varepsilon\, c \wedge c \,\varepsilon\, b))$

4.4.2 $\subset$–axioms

Lejewski (Lejewski, [24, p. 137]) introduced an axiom:

Axiom 5 $\forall_{a,b}[a \,\overline{\varepsilon}\, b \equiv (\exists_c(c \,\overline{\varepsilon}\, b \wedge \neg(c \,\overline{\varepsilon}\, a)) \wedge \forall_{c,d,e}(c \,\overline{\varepsilon}\, d \wedge d \,\overline{\varepsilon}\, e \rightarrow c \,\overline{\varepsilon}\, a \vee d \,\overline{\varepsilon}\, a))]$

where the functor ε was introduced by the following definition:

Definition 35 $\forall_{a,b}(a \,\varepsilon\, b \equiv \exists_c(a \,\overline{\varepsilon}\, c \wedge \neg(a \,\overline{\varepsilon}\, b)))$

Functor $\overline{\varepsilon}$ is called *the functor of singular exclusion.* First, on the ground of formulations of Ontology, where ε was the primitive functor, it was defined:

Definition 36 $\forall_{a,b}(a \,\overline{\varepsilon}\, b \equiv (a \,\varepsilon\, a \wedge \neg(a \,\varepsilon\, b)))$

Sobociński's result tells us that we can accept the functor $<$ as the only primitive one functor of Ontology. This functor originally was introduced by the definition:

Definition 37 $\forall_{a,b}[a < b \equiv (\exists_c c\ \varepsilon\ a \wedge \forall_c(c\ \varepsilon\ a \rightarrow c\ \varepsilon\ b))]$

In Sobociński's formulation however, it was the only primitive functor. The axiom was:

Axiom 6 $\forall_{a,b}[a < b \equiv (\exists_c c < a \wedge \forall_c(c < a \wedge \forall_d(d < c \rightarrow c < d) \rightarrow c < b))]$

and the functor ε was introduced by the definition:

Definition 38 $\forall_{a,b}(a\ \varepsilon\ b \equiv (a < b \wedge \forall_c(c < a \rightarrow a < c)))$

In 1956 Lejewski has shown that we can use $\subset$ as the only primitive functor. In the original formulation it was defined:

Definition 39 $\forall_{a,b}(a \subset b \equiv \forall_c(c\ \varepsilon\ a \rightarrow c\ \varepsilon\ b))$

In Lejewski's formulation, ε was defined:

Definition 40 $\forall_{a,b}[a\ \varepsilon\ b \equiv (\exists_c \neg(a \subset c) \wedge a \subset b \wedge \forall_{c,d,e}(c \subset a \wedge d \subset a \rightarrow$
$\rightarrow (c \subset d \vee d \subset e)))]$

and Lejewski's axiom (the second, shorter version) was:

Axiom 7 $\forall_{a,b}[a \subset b \equiv \forall_c(c \subset a \wedge \forall_{d,e}(d \subset c \rightarrow c \subset d \vee d \subset e) \rightarrow c \subset b)]$

4.5 Rules

In different formulations of Ontology we can meet different bases of primitive rules of inference. For instance, Luschei (Luschei, [31, p. 141]) enumerates: rule(s) of definition, distribution, detachment, substitution and extensionality. On the other hand, Słupecki (Słupecki, [42, pp. 75-79]) equivalently lists: substitution, detachment, omitting the universal quantifier, omitting the particular quantifier[47], adding the universal quantifier, adding the particular quantifier, the rule of definition, and the rule of extensionality.

In our investigations we will be interested whether the above mentioned rules are validity-preserving in non-standard interpretations of primitive functors.

In the presentation of rules, we shall follow the Słupecki's account.

[47]Słupecki calls it 'existential'. For the reasons already given, we choose another name.

Rule of Substitution The original Leśniewski's account of this rule is a little bit foreign to the modern logical language. His description of this rule uses numerous technical terms of his own. We'll try to give a simplified formulation of this rule, which (when given with the rules of adding and omitting quantifiers) does the same job as the original one.[48]

> Let χ be a propositional expression with at least one free variable ν of syntactic category σ. From χ infer an expression χ_1 which differs from χ in the following aspects:
>
> 1. Instead of each free occurrence of ν in χ there is an expression (simple or not) μ of the category σ.
> 2. It is not the case that there is in μ a free variable υ and ν occurs in χ as a free variable within a range of a quantifier binding υ.

In other words, we substitute for each free occurrence of a variable so that no variable becomes bound.

Theorem 3 *The rule of substitution is validity-preserving in BL, independently of the choice of the interpretation of a primitive functor.*

PROOF: In BL the only variables which can be substituted for are name variables. They can be substituted either by name variables or by name constants, as far as the latter have been introduced by means of the rule of definition.

The semantics for name variables has been introduced. Semantics of name constants is very intuitive: they denote for each OBJ a foxed subset of

[48] Let's take the Luschei's translation of this description (Luschei, [31, pp. 252-253]). It has 14 points. For the sake of example, I quote the 3rd and 9th:

> (3) For every E and F, if E is d, F is a word inside the subquantifier of C, and there are exactly as many d that precede E as words inside the subquantifier of C that precede F, then, for at least one G, F is a variable of C bound by G or is an expression equiform to E.
>
> (9) For every E, if E is a term in A, then either (i) E is an argument or functor of an ingredient of A and suited to be constant, relative to thesis B of this system; (ii) for at least one F, E is a word inside F, and F is a universal quantifier in A; or (iii) for certain F and G, F is in A, and E is a variable of generalization F bound by G.

Clearly, a rule given in 14 points of similar style is not user-friendly.

a domain, and this denotation does not depend on the interpretation of variables.

Now assume that a given expression χ of BL is valid and ν is a free name variable in χ. Then, obviously, χ is satisfied in any model; in any domain and in any valuation, particularly in any interpretation of ν.

Now, let χ_1 be a result of replacing ν in χ by μ. Since χ_1 differs from χ only in the form of one free variable, the conditions of validity of χ_1 differ only in the fact that we talk about all possible interpretations of μ instead of that of ν. In any model, possible different interpretations of μ are exactly the same as possible interpretations of ν - namely all subsets of a model's domain. Hence, χ_1 is also valid.

In the case of substituting a name variable by a name constant, the case is even simpler. In general, if a formula is valid in any possible interpretation of a given variable ν, in fact, it yields a valid result if we admit only one interpretation of this variable (the intended denotation of the name we want to substitute).

Q.E.D.

Remark It is an interesting, how the above theorem can be extended to the full language of Ontology. Let us sketch a possible and quite a simple way of doing so, on the example of variables and constants of the category $\frac{s}{n,n}$.

If we want to consider variables and constants of this category, it will be convenient to introduce their 'denotation' or 'valuation'. Name constants have some fixed subsets of OBJ as denotations. Name variables have subsets of OBJ as possible valuations. Accordingly, intuitive candidates for our job will be binary relations between the subsets of OBJ. Strictly speaking, not all possible relations, but only the relations corresponding to *definientes* of possible definitions of ontological functors, when a semantic interpretation of the primitive basis is given.[49]

For example, the relation corresponding to the standard interpretation of ε is the relation that takes place between two subsets of a given OBJ iff the first is a singleton and a subset of the second. Relations corresponding to other, non-primitive functors are to be settled in accordance with their definitions in the system.

Clearly, OF-constants in a given model OBJ have fixed relations in 2^{OBJ} as semantic correlates. Valuations of OF-variables of a given language in a

[49] For details concerning possible definitions of ontological functors etc. see (Urbaniak, [47]).

given model will be sequences of (admissible) relations in 2^{OBJ}. Semantic interpretation of quantifiers binding OF-variables is intuitive — it repeats the schema for name variables, differing only in the set of admissible valuations of OF-variables. Now, the proof that the rule of substitution for such variables is validity-preserving should work in a similar manner as that for name-variables.

Full language of Ontology is not restricted only to these two kinds of variables and constants. It admits variables of any category built from n-s or s-s and any constants of any admissible category, if these constants are introduced by properly constructed definitions.

It seems that *mutatis mutandis* we can extend our semantics to let it account for any admissible category (perhaps it would be then convenient to introduce explicitly 1 and 0 as semantic correlates of propositions). These are only technical details.[50] Hence:

Hypothesis 1 *The rule of substitution is validity-preserving in the full language of Ontology.*

Rules for quantification The rules governing the use of quantifiers are very similar to that occurring in systems of natural deduction — for instance, in the formulation introduced in (Borkowski, [7]).

Their validity[51] is based, freely speaking, on the relations between quantification, free variables and semantic correlates of expressions of a given category, just as it is in first- or second-order predicate logic. Interpretation of OF-constants does not interfere with their validity, hence:

Hypothesis 2 *Rules governing the use of quantifiers are validity-preserving independently on the interpretation of OF-constants.*

Now, we will simply present the rules of definition and of extensionality. Later on, we shall try to answer the question whether these rules preserve validity in given non-standard interpretations.

[50]Though it is interesting question, how to construct quite a simple semantics for the full language of Ontology, which (semantics) would not only do its formal job, but also would work in accordance with Leśniewski's philosophical commitment, i.e. would not introduce other entities than individuals.

[51]When there is no danger of ambiguity, we use 'validity' with reference to rules as a name of the property of being validity-preserving.

Rule of extensionality This rule is an axiomatic rule introducing implications of a specific form (so called *laws of extensionality*) to a system irrespective of what theorems were subjoined thereto before. In its presentation we shall follow (Słupecki, [42, p. 101]).

> Let the variable χ represent propositional function (complex or simple — what is important is the category of χ and not its complexity) of one argument, and let the not equiform variables λ and δ be admissible arguments of this function. Let $\chi(\lambda)$ and $\chi(\delta)$ be applications of χ to, respectively, λ and δ. The consequents of the laws of extensionality have the form:
>
> $$\forall_\chi[\chi(\lambda) \equiv \chi(\delta)]$$
>
> The form of the antecedent depends on the category of λ and δ. If they are nominal variables not having the form of the variable x, the antecedent has the form:
>
> $$\forall_x[x \varepsilon \lambda \equiv x \varepsilon \delta]$$
>
> If λ and δ are functors of k arguments, and $\alpha_1, \ldots, \alpha_k$ are (i) arguments of both λ and δ, (ii) none two of them are equiform, and (iii) none has the form of the variable x, then we consider two possibilities.
>
> 1. Both λ and δ are name-formative. Then, the antecedent has the form:
>
> $$\forall_{x,\alpha_1,\ldots,\alpha_k}[x \varepsilon \lambda(\alpha_1, \ldots, \alpha_k) \equiv x \varepsilon \delta(\alpha_1, \ldots, \alpha_k)]$$
>
> 2. Both λ and δ are proposition-formative. Then, the antecedent has the form:
>
> $$\forall_{\alpha_1,\ldots,\alpha_k}[\lambda(\alpha_1, \ldots, \alpha_k) \equiv \delta(\alpha_1, \ldots, \alpha_k)]$$

Rule of Definition We shall first present this rule for nominal constants and name-formative or proposition-formative functors of nominal arguments according to (Słupecki, [42, pp. 74-75]), and then extend it to the full language of Ontology, following (Słupecki, [42, pp. 100-101]).

1. The schema for defining a nominal constant, say α, is as follows:

$$\forall_b(b \,\varepsilon\, \alpha \equiv b \,\varepsilon\, b \wedge \chi)$$

   where χ is a propositional expression of BL.

2. The schema for a definition of a proposition-formative functor, say λ, of nominal arguments is:

$$\lambda(b_1, \ldots, b_n) \equiv \chi$$

   where χ is as before.

3. The schema for defining a name-formative functor, say δ, is as follows:

$$b \,\varepsilon\, \delta(b_1, \ldots, b_n) \equiv b \,\varepsilon\, b_i \wedge \chi$$

   where $0 \leq i \leq n$ and χ is as above.

In definitions other than these above only functors are defined. These functors are either proposition-formative or name-formative. The forms of definitions do not differ essentially from the above schemata, but the functors of the full language of Ontology may depend on parameters. These parameters are placed inside parentheses differing in shape from those enclosing arguments of functors. For every variable acting as a parameter of a functor, the definiens of the definition of this functor has to include a free variable equiform to that occurring in the parameter, and every variable acting as a parameter differs in form from all other variables appearing in the definiendum. The rule of substitution may affect parameters as well.

This rule, even for BL, is a cause of problems which throw some light on some interesting issues, which will be discussed later on.

4.6 Adherents of Standard Semantics

The Standard Semantics of Ontology consists in accepting the satisfaction $^{\varepsilon}$ as the notion of satisfaction for the ε functor. This interpretation of ε consists in the following underlying intuition:

> Let a, b be names. The expression $a \varepsilon b$ is true *if and only if* a is an unshared name (i.e. having exactly one designate) and, either b is an unshared name of the object named by a, or b is a shared name (i.e. having more than one designate) naming also the object named by a.

This interpretation is accepted by numerous logicians.[52]

What is more interesting, almost all of them explicitly claim that this meaning of ε functor is determined by the Axiom 1 ((Słupecki, [42, p. 72–73]), (Lejewski, [24, p. 135–136]), (Hiż, [17, p. 273])); perhaps (Iwanuś, [18, p. 168-169]) is the most exemplary:

> The only primitive term of ontology is the constant ' ε ', its meaning is determined by the following axiom ...

similarly, (Lejewski, [26, p. 323]):

> ...the meaning of the copula 'is' (' ε ' in symbols) is determined axiomatically...

What does it mean that an axiom determines the meaning of the primitive constant? Such a claim is an equivalence, which states that

1. If we accept the 'determined' meaning of the constant, the axiom is valid.
2. If the axiom is valid, then the constant has the such 'determined' meaning.

In the first point the mentioned adherents are right, in the sense, that in the Standard Semantic the Axiom 1 (and other axioms) are valid $^{\varepsilon}$ in the standard interpretation of ε .

Our purpose is to show that they are wrong in the second claim — we shall argue that we can understand ε in at least $\aleph_0$ different ways and keep axioms valid in this interpretations.

We must also point out that it is not clear, whether the expression 'determined by an axiom' means the same as 'determined axiomatically'. The second expression can be quite legitimately interpreted as 'determined by axiomatic basis, i.e. by axiom(s) and rule(s) of a given system'. In the second sense, we must also consider additional two conditions.

[52] E.g.: (Borkowski, [7, p. 188]), (Sobociński, [43, p. 14]), (Lejewski, [25, p. 54–55]), (Lejewski, [24, p. 129]) , (Słupecki, [42, p. 65–66]), (Canty, [8, p. 149–151]).

1. If we attribute to a constant the standard interpretation, all rules preserve validity.

2. If the validity of rules is to be preserved, the constant has to have the standard interpretation.

In other words, for a given axiomatic basis and for any non-standard interpretation we will be interested not in one, but in two questions. First, whether its axiom is valid in this interpretation of the primitive constant, and second, whether all rules preserve validity in this interpretation of this constant.[53]

Adherents of the uniqueness of the admissible interpretation of ε can remain unconvinced when they are shown a non-standard interpretation which preserves axiom(s). However, non-standard interpretations which not only keep axiom(s) valid, but also make rules validity-preserving, may be for them really troublesome.

We shall show that there are non-standard interpretations which keep axiom(s) safe (it is our first purpose), and also we shall discuss, whether in these interpretations rules of Ontology are validity-preserving.

4.7 Non-standard Validity of ε —Axioms

4.7.1 Satisfaction$^\Phi$ and ε –axioms

Let us instead of $satsifaction^{\varepsilon}$ accept the following definition of satisfaction:

Definition 41 *We assume that the sequence of variables is fixed.*

1. $\lceil \mu_k \, \varepsilon \, \mu_i \rceil$ *is satisfied*$^\Phi$ *in* OBJ *by a valuation* A^u if and only if $A_k = A_i = \Phi$.

2. $\lceil \neg\chi \rceil$ *is satisfied*$^\Phi$ *in* OBJ *by the valuation* A^u if and only if $\lceil \chi \rceil$ *is not satisfied*$^\Phi$ *in* OBJ *by the valuation* A^u.

3. $\lceil \chi_i \wedge \chi_j \rceil$ *is satisfied*$^\Phi$ *in* OBJ *by the valuation* A^u if and only if $\lceil \chi_i \rceil$ *is satisfied*$^\Phi$ *in* OBJ *by the valuation* A^u *and* $\lceil \chi_j \rceil$ *is satisfied*$^\Phi$ *in* OBJ *by the valuation* A^u.

[53] Note that, though we discuss soundness in non-standard interpretations. It is an interesting question which we shall not discuss, whether Ontology is a sound *and complete* theory in some non-standard interpretation.

4. $\ulcorner\forall_{\mu_k}\chi\urcorner$ *is satisfied*$^{\Phi}$ *in* OBJ *by the valuation* A^u if and only if $\ulcorner\chi\urcorner$ *is satisfied*$^{\Phi}$ *in* OBJ *by any possible valuation* A^d *which is different from* A^u *at most on one place, namely* $k-th$ *(place).*

As an obvious consequence of the standard definition of particular quantifier, we may add:

> $\ulcorner\exists_{\mu_k}\chi\urcorner$ is satisfied$^{\Phi}$ in OBJ by a valuation A^u *if and only if* $\ulcorner\chi\urcorner$ is satisfied$^{\Phi}$ in OBJ by some possible valuation A^d which is different from A^u at most on one place, namely $k-th$ (place).

Theorem 4 *Axiom 1 is valid*$^{\Phi}$*.*

PROOF: *Implication to the right*: We assume that $a\,\varepsilon\,b$ is $SAT^{\Phi,u}_{OBJ}$. It is to be proven, that

$$\exists_c c\,\varepsilon\,a \wedge \forall_{c,d}(c\,\varepsilon\,a \wedge d\,\varepsilon\,a \rightarrow c\,\varepsilon\,d) \wedge \forall_c(c\,\varepsilon\,a \rightarrow c\,\varepsilon\,b)$$

is $SAT^{\Phi,u}_{OBJ}$.

Since $a\,\varepsilon\,b$ is $SAT^{\Phi,u}_{OBJ}$, $V^u(a) = V^u(b) = \Phi$. There is a valuation A^s which differs from A^u at most in the fact that the $V^s(c) = \Phi$ such that $c\,\varepsilon\,a$ is $SAT^{\Phi,s}_{OBJ}$. Hence, $\exists_c c\,\varepsilon\,a$ is $SAT^{\Phi,u}_{OBJ}$.

Now we shall show that with this assumptions

$$\forall_{c,d}(c\,\varepsilon\,a \wedge d\,\varepsilon\,a \rightarrow c\,\varepsilon\,d)$$

is $SAT^{\Phi,u}_{OBJ}$. Let us assume that $c\,\varepsilon\,a \wedge d\,\varepsilon\,a$ is $SAT^{\Phi,u}_{OBJ}$. Since, as we have said, $V^u(a) = \Phi$, it must be the case that also $V^u(c) = V^u(d) = \Phi$. If this is the case, also $c\,\varepsilon\,d$ is $SAT^{\Phi,u}_{OBJ}$.

It remains to show that on the ground of our assumptions,

$$\forall_c(c\,\varepsilon\,a \rightarrow c\,\varepsilon\,b)$$

is $SAT^{\Phi,u}_{OBJ}$. To do so, assume that $c\,\varepsilon\,a$ is $SAT^{\Phi,u}_{OBJ}$. Then, obviously, $V^u(c) = V^u(a) = \Phi$. Since, according to our previous assumption, $V^u(b) = \Phi$, we obtain $V^u(c) = V^u(b) = \Phi$. It follows that $c\,\varepsilon\,b$ is $SAT^{\Phi,u}_{OBJ}$, what ends the proof in one direction.

Implication to the left: We assume that

$$\exists_c c\,\varepsilon\, a \wedge \forall_{c,d}(c\,\varepsilon\, a \wedge d\,\varepsilon\, a \rightarrow c\,\varepsilon\, d) \wedge \forall_c(c\,\varepsilon\, a \rightarrow c\,\varepsilon\, b)$$

is $SAT^{\Phi,u}_{OBJ}$, and obtain in consequence that $a\,\varepsilon\, b$ is $SAT^{\Phi,u}_{OBJ}$.

Since $\exists_c c\,\varepsilon\, a$, there is a valuation A^s differing at most in the value assigned to c such that $c\,\varepsilon\, a$ is $SAT^{\Phi,s}_{OBJ}$. Therefore, $V^s(c) = V^s(a) = \Phi$ Since A^s differs from A^u only in $V^s(c)$, it is the case that $V^u(a) = \Phi$. Let us now notice that $\forall_c(c\,\varepsilon\, a \rightarrow c\,\varepsilon\, b)$ is $SAT^{\Phi,u}_{OBJ}$. As we have said, there is a valuation A^s differing at most in the value assigned to c such that $c\,\varepsilon\, a$ is $SAT^{\Phi,s}_{OBJ}$. Hence, $c\,\varepsilon\, b$ is $SAT^{\Phi,s}_{OBJ}$. It means that $V^s(c) = V^s(b) = \Phi$. But A^s differs from A^u at most in the value assigned to c. Therefore $V^s(b) = V^u(b) = \Phi$. Hence $V^u(a) = V^u(b) = \Phi$. It means that $a\,\varepsilon\, b$ is $SAT^{\Phi,u}_{OBJ}$, what ends the proof (if we take under consideration that we can repeat our proof for any OBJ and any valuation). Q.E.D.

The proof given above may be stated in a more intuitive, but less precise and not quite exact manner.

Implication to the right: Assume $a\,\varepsilon\, b$. Hence, both a and b are empty names. Therefore, $b\,\varepsilon\, a$. If it is so, then also $\exists_c c\,\varepsilon\, a$.

Now assume $c\,\varepsilon\, a \wedge d\,\varepsilon\, a$. then, both c and d are empty. Hence $c\,\varepsilon\, d$.

Therefore, $\forall_{c,d}(c\,\varepsilon\, a \wedge d\,\varepsilon\, a \rightarrow c\,\varepsilon\, d)$.

Assume $c\,\varepsilon\, a$. Then c is empty. As we have said, b is also empty. therefore also $c\,\varepsilon\, b$.

So, $\forall_c(c\,\varepsilon\, a \rightarrow c\,\varepsilon\, b)$.

Implication to the left: Assume

$$\exists_c c\,\varepsilon\, a \wedge \forall_{c,d}(c\,\varepsilon\, a \wedge d\,\varepsilon\, a \rightarrow c\,\varepsilon\, d) \wedge \forall_c(c\,\varepsilon\, a \rightarrow c\,\varepsilon\, b)$$

If $\exists_c c\,\varepsilon\, a$, it means that some name is empty together with a. So a is empty.

Since $\forall_c(c\,\varepsilon\, a \rightarrow c\,\varepsilon\, b)$, every name which is empty together with a is empty together with b. If a is empty together with itself, a is empty together with b. In fact, aea; hence $a\,\varepsilon\, b$, what ends the proof.

What we would like to show is that it does not matter which axiom we choose, our non-standard interpretation of ε makes this axiom valid.

One could argue that we do not have to prove it independently for each axiom. It has been already shown that those axioms are on the ground of

Ontology equivalent, so the first proof suffice for our purpose. Nevertheless, we would like to obtain our results independently of the results concerning rules of Ontology.

Theorem 5 *Axiom 2 is valid*$^{\Phi}$.

PROOF: *Implication to the right*: Assume that $a \varepsilon b$ is $SAT^{\Phi,u}_{OBJ}$. Then, $V^u(a) = V^u(b) = \Phi$. There is the valuation A^s which differs from A^u at most in the fact that $V^s(c) = \Phi$ such that the formula $c \varepsilon a \wedge c \varepsilon b$ is $SAT^{\Phi,s}_{OBJ}$. Since A^u differs from A^s only in the valuation of c, $\exists_c(c \varepsilon a \wedge c \varepsilon b)$ is $SAT^{\Phi,u}_{OBJ}$.

Let us now assume that $c \varepsilon a \wedge d \varepsilon a$ is $SAT^{\Phi,u}_{OBJ}$. This means that $V^u(c) = V^u(a) = V^u(d) = \Phi$. Obviously, then $c \varepsilon d$ is $SAT^{\Phi,u}_{OBJ}$.

Therefore, $\forall_{c,d}(c \varepsilon a \wedge d \varepsilon a \rightarrow c \varepsilon d)$ is $SAT^{\Phi,u}_{OBJ}$.

Implication to the left: Assume

$$\exists_c(c \varepsilon a \wedge c \varepsilon b) \wedge \forall_{c,d}(c \varepsilon a \wedge d \varepsilon a \rightarrow c \varepsilon d)$$

is $SAT^{\Phi,u}_{OBJ}$.

If $\exists_c(c \varepsilon a \wedge c \varepsilon b)$ is $SAT^{\Phi,u}_{OBJ}$, then there is a valuation A^s differing from A^u at most in the value assigned to c such that $c \varepsilon a \wedge c \varepsilon b$ is $SAT^{\Phi,s}_{OBJ}$. Therefore, $V^s(c) = V^s(a) = V^s(b) = V^u(a) = V^u(b) = \Phi$. Hence, $a \varepsilon b$ is $SAT^{\Phi,u}_{OBJ}$, what (since we can reason in the same way for any model and any valuation) ends the proof. Q.E.D.

Of course, there is also the 'light' version of this reasoning.

Implication to the right: Assume $a \varepsilon b$. Then a and b are empty. Since, a is empty together with itself, $a \varepsilon a$. Hence, $a \varepsilon a \wedge a \varepsilon b$. Therefore, $\exists_c(c \varepsilon a \wedge c \varepsilon b)$.

Now assume $c \varepsilon a \wedge d \varepsilon a$. Then, obviously, a, c, d are together empty. Hence $c \varepsilon d$.

Implication to the left: Assume

$$\exists_c(c \varepsilon a \wedge c \varepsilon b) \wedge \forall_{c,d}(c \varepsilon a \wedge d \varepsilon a \rightarrow c \varepsilon d)$$

Since $\exists_c(c \varepsilon a \wedge c \varepsilon b)$ there is a name which is together empty with a and b. Therefore, a and b are both empty. Hence $a \varepsilon b$, what ends the reasoning.

Theorem 6 *Axiom 3 is valid*$^{\Phi}$.

Proof: *Implication to the right*: Assume $a \varepsilon b$ is $SAT^{\Phi,u}_{OBJ}$. Hence $V^u(a) = V^u(b) = \Phi$. There is a valuation A^s which differs from A^u at most in the fact that $V^s(c) = \Phi$. The formula $c \varepsilon a \wedge c \varepsilon b$ is $SAT^{\Phi,s}_{OBJ}$. Since A^u differs from A^s only in the valuation of c, $\exists_c(c \varepsilon a \wedge c \varepsilon b)$ is $SAT^{\Phi,u}_{OBJ}$.

Assume moreover that for some k, $c \varepsilon a$ is $SAT^{\Phi,k}_{OBJ}$. It means that $V^k(c) = V^k(a) = \Phi$. It is obvious that then $a \varepsilon c$ is $SAT^{\Phi,k}_{OBJ}$.

Hence $\forall_c(c \varepsilon a \rightarrow a \varepsilon c)$ is $SAT^{\Phi,\psi}_{OBJ}$.

Implication to the left: Assume

$$\exists_c(c \varepsilon a \wedge c \varepsilon b) \wedge \forall_c(c \varepsilon a \rightarrow a \varepsilon c)$$

is $SAT^{\Phi,u}_{OBJ}$. If $\exists_c(c \varepsilon a \wedge c \varepsilon b)$ is $SAT^{\Phi,u}_{OBJ}$, then there is a valuation A^s differing from A^u at most in the value assigned to c such that $c \varepsilon a \wedge c \varepsilon b$ is $SAT^{\Phi,s}_{OBJ}$. Therefore, $V^s(c) = V^s(a) = V^s(b) = V^u(a) = V^u(b) = \Phi$. Hence, $a \varepsilon b$ is $SAT^{\Phi,u}_{OBJ}$, what (since we can reason in the same way for any model and any valuation) ends the proof. Q.E.D.

Theorem 7 *Axiom 4 is valid*$^{\Phi}$.

Proof: *Implication to the right*: Assume that $a \varepsilon b$ is $SAT^{\Phi,u}_{OBJ}$. Hence $V^u(a) = V^u(b) = \Phi$. There is a valuation A^s which differs from A^u at most in the fact that $V^s(c) = \Phi$. The formula $a \varepsilon c \wedge c \varepsilon b$ is $SAT^{\Phi,s}_{OBJ}$. Since A^u differs from A^s only in the valuation of c, $\exists_c(a \varepsilon c \wedge c \varepsilon b)$ is $SAT^{\Phi,s}_{OBJ}$.

Implication to the left: Assume that $\exists_c(a \varepsilon c \wedge c \varepsilon b)$ is $SAT^{\Phi,u}_{OBJ}$. Hence, there is a valuation A^s which differs from A^u at most in the fact that $V^s(c) = \Phi$, such that $a \varepsilon c \wedge c \varepsilon b$ is $SAT^{\Phi,s}_{OBJ}$. If it is so, $V^s(a) = V^s(c) = V^s(b) = \Phi$ Since, as we have said, A^s differs from A^u at most in the fact that $V^s(c) = \Phi$, it is also the case that $V^u(a) = V^u(b) = \Phi$. Therefore, $a \varepsilon b$ is $SAT^{\Phi,u}_{OBJ}$. We can repeat this reasoning for any OBJ and any A^u, what ends the proof.

Q.E.D.

4.7.2 Satisfaction^F and ε –axioms

Let us instead of $satsifaction^{\Phi}$ interpret ε as *falsum* functor, which always yields 0 as value for any arguments. Therefore we accept the following definition of satisfaction^F:

Definition 42 *We assume that the sequence of variables is fixed.*

1. $\lceil \mu_k \, \varepsilon \, \mu_i \rceil$ *is satisfied*^{F} *in* OBJ *by a valuation* A^u if and only if $A_k \neq A_k$.

2. $\lceil \neg\chi \rceil$ *is satisfied*^{F} *in* OBJ *by the valuation* A^u if and only if $\lceil \chi \rceil$ *is not satisfied*^{F} *in* OBJ *by the valuation* A^u.

3. $\lceil \chi_i \wedge \chi_j \rceil$ *is satisfied*^{F} *in* OBJ *by the valuation* A^u if and only if $\lceil \chi_i \rceil$ *is satisfied*^{F} *in* OBJ *by the valuation* A^u *and* $\lceil \chi_j \rceil$ *is satisfied*^{F} *in* OBJ *by the valuation* A^u.

4. $\lceil \forall_{\mu_k} \chi \rceil$ *is satisfied*^{F} *in* OBJ *by the valuation* A^u if and only if $\lceil \chi \rceil$ *is satisfied*^{F} *in* OBJ *by any possible valuation* A^d *which is different from* A^u *at most on one place, namely* $k - th$ *(place).*

As an obvious consequence of the standard definition of particular quantifier, we may add:

> $\lceil \exists_{\mu_k} \chi \rceil$ is satisfied^F in OBJ by the valuation A^u *if and only if* $\lceil \chi \rceil$ is satisfied^F in OBJ by some possible valuation A^d which is different from A^u at most on one place, namely $k - th$ (place).

Theorem 8 *Axiom 1 is valid*^{F}.

PROOF: *Implication to the right*: Assume that $a \, \varepsilon \, b$ is $SAT^{F,u}_{OBJ}$. Since, from the semantic interpretation of ε, $a \, \varepsilon \, b$ cannot be $SAT^{F,u}_{OBJ}$, we, by $p \wedge \neg p \rightarrow q$, obtain the right side of the equivalence.

Implication to the left: Assume that the conjunction on the right side is $SAT^{F,u}_{OBJ}$. Hence, $\exists_c c \, \varepsilon \, a$ is $SAT^{F,u}_{OBJ}$. Therefore, there exists the valuation A^s that differs from A^u at most in the assignment for c, such that $c \, \varepsilon \, a$ is $SAT^{F,s}_{OBJ}$. Since it cannot occur, according to our interpretation of ε, by $p \wedge \neg p \rightarrow q$ we obtain the left side of the equivalence. The reasoning is repeatable for any model and any valuation. Q.E.D.

Theorem 9 *Axiom 2 is valid*^{F}.

PROOF: *Implication to the right*: Assume $a \, \varepsilon \, b$ is $SAT^{F,u}_{OBJ}$. But $a \, \varepsilon \, b$ cannot be $SAT^{F,u}_{OBJ}$. Hence, right side of the equivalence in the axiom also is

$SAT^{F,u}_{OBJ}$(since — in classical logic at least — contradiction implies any proposition).

Implication to the left: Assume $\exists_c(c \varepsilon a \wedge c \varepsilon b)$ is $SAT^{F,u}_{OBJ}$. Therefore, $c \varepsilon a \wedge c \varepsilon b$ is satisfied^F in OBJ by a valuation A^s which differs from A^u at most on one place, namely in $V^s(c)$. Therefore, $c \varepsilon a$ is $SAT^{F,u}_{OBJ}$. But it cannot be satisfied. From this contradiction, the left side of the equivalence results. Q.E.D.

Theorem 10 *Axiom 3 is valid*^{F}*.*

The proof is exactly the same as in the case of Theorem 9.

Theorem 11 *Axiom 4 is valid*^{F}*.*

PROOF: The proof of implication to the right is the same, as in the proof of Theorem 9.

Implication to the left: Assume that $\exists_c(a \varepsilon c \wedge c \varepsilon b)$ is $SAT^{F,u}_{OBJ}$. Obviously, there is a valuation A^s which differs from A^u only in the value assigned co c, such that $a \varepsilon c \wedge c \varepsilon b$ is $SAT^{F,s}_{OBJ}$. Hence, $a \varepsilon c$ is $SAT^{F,s}_{OBJ}$. But it cannot be the case. Therefore, from contradiction, we obtain the left side of the equivalence. Q.E.D.

4.8 Non-standard Interpretations of $\subset$–axioms

4.8.1 Axiom 5

In this axiom the primitive term is $\overline{\varepsilon}$ –the functor of singular exclusion. In standard interpretation, intuitively, it yields true statement with its arguments a, b iff a is an unshared name, and b does not name the object named by a (it does not matter, whether b is unshared, shared, or empty).

Formally, we can accommodate the definition of satisfaction $^{\varepsilon}$ to make $\overline{\varepsilon}$ the primitive term. Instead of first condition of the definition of satisfaction $^{\varepsilon}$, we simply introduce:

Definition 43 $\lceil \mu_k \overline{\varepsilon} \mu_i \rceil$ *is satisfied*$^{\varepsilon}$ *in* OBJ *by a valuation* A^u *iff* $\exists!_x x \in A_k \wedge \neg\exists_y[y \in A_k \wedge y \in A_i]$

We will now show that the Axiom 5 is valid also in some unintended interpretations of $\overline{\varepsilon}$.

According to the Definition 36, $a \overline{\varepsilon} b$ is (in systems with ε as primitive functor) defined by means of $a \varepsilon a \wedge \neg(a \varepsilon b)$. Let us follow this intuition, using the notion of satisfaction$^\Phi$. We obtain:

Definition 44 *$\mu_k \overline{\varepsilon} \mu_i$ is satisfied$^\Phi$ in OBJ by A^u iff $\mu_k \varepsilon \mu_k \wedge \neg(\mu_k \varepsilon \mu_i)$ is satisfied$^\Phi$ in OBJ by A^u.*

In terms of model theory, it is equivalent to:

Definition 45 *$\mu_k \overline{\varepsilon} \mu_i$ is satisfied$^\Phi$ in OBJ by A^u iff $V^u(\mu_k) = \Phi \wedge V^u(\mu_i) \neq \Phi$.*

In an intuitive manner: μ_k is empty and μ_i is not empty.

Theorem 12 *Axiom 5 is valid$^\Phi$.*

PROOF: *Implication to the right*: Assume $a \overline{\varepsilon} b$ is $SAT^{\Phi,u}_{OBJ}$. Then, $V^u(a) = \Phi \wedge V^u(b) \neq \Phi$.

There exists a valuation A^s which differs from A^u at most in the fact, that $V^s(c) = \Phi$, such that $c \overline{\varepsilon} b \wedge \neg(c \overline{\varepsilon} a)$ is $SAT^{\Phi,s}_{OBJ}$. It is so, because $V^s(c) = \Phi$, $V^s(b) \neq \Phi$, and it is not the case that both: $V^s(c) = \Phi$ and $V^s(a) \neq \Phi$.[54]

Hence, $\exists_c(c \overline{\varepsilon} b \wedge \neg(c \overline{\varepsilon} a))$ is $SAT^{\Phi,u}_{OBJ}$.

Now, assume additionally that $c \overline{\varepsilon} d \wedge d \overline{\varepsilon} e$ is $SAT^{\Phi,u}_{OBJ}$. It means that Φ , $V^u(d) \neq \Phi$, $V^u(d) = \Phi$, and $V^u(e) \neq \Phi$. Obviously, we have a contradiction. Hence, we obtain by $p \wedge \neg p \rightarrow q$ that semantic conditions of satisfaction$^\Phi$ of $c \overline{\varepsilon} a \vee d \overline{\varepsilon} a$ are fulfilled, and $c \overline{\varepsilon} a \vee d \overline{\varepsilon} a$ is $SAT^{\Phi,u}_{OBJ}$.

Implication to the left: Assume

$$\exists_c(c \overline{\varepsilon} b \wedge \neg(c \overline{\varepsilon} a)) \wedge \forall_{c,d,e}(c \overline{\varepsilon} d \wedge d \overline{\varepsilon} e \rightarrow d \overline{\varepsilon} a)$$

is $SAT^{\Phi,u}_{OBJ}$.

Since $\exists_c(c \overline{\varepsilon} b \wedge \neg(c \overline{\varepsilon} a))$ is $SAT^{\Phi,u}_{OBJ}$, there exists a valuation A^s differing from A^u at most in the value assigned to c, such that $c \overline{\varepsilon} b \wedge \neg(c \overline{\varepsilon} a)$ is $SAT^{\Phi,s}_{OBJ}$. It means that $V^s(c) = \Phi$, $V^s(b) \neq \Phi$. Moreover, it implies that it is not the case, that both: $V^s(c) = \Phi$ and $V^s(a) \neq \Phi$. Since $V^s(c) = \Phi$, obviously $V^s(a) = \Phi$. According to our understanding of A^s,

[54] Since $V^u(a) = V^s(a)$ and $V^u(b) = V^s(b)$.

$V^u(a) = V^s(a)$ and $V^u(b) = V^s(b)$. Therefore, $V^u(b) \neq \Phi$ and $V^u(a) = \Phi$. Therefore, $a \,\overline{\varepsilon}\, b$ is $SAT^{\Phi,s}_{OBJ}$. This reasoning can be repeated for any OBJ and any A^u. This ends the proof. Q.E.D.

We can also follow the Definition 36 using the notion of satisfaction^F. We obtain:

Definition 46 *$a \,\overline{\varepsilon}\, b$ is satisfied^F in OBJ by A^u iff $V^u(a) \neq V^u(a)$.*

Theorem 13 *Axiom 5 is valid^F.*

PROOF: *Implication to the right*: Assume $a \,\overline{\varepsilon}\, b$ is $SAT^{F,u}_{OBJ}$. Obviously, it cannot be satisfied. From this contradiction, we obtain the right side of the equivalence.

Implication to the left: Assume

$$\exists_c (c \,\overline{\varepsilon}\, b \wedge \neg(c \,\overline{\varepsilon}\, a)) \wedge \forall_{c,d,e} (c \,\overline{\varepsilon}\, d \wedge d \,\overline{\varepsilon}\, e \rightarrow d \,\overline{\varepsilon}\, a)$$

is $SAT^{F,u}_{OBJ}$.

Since $\exists_c (c \,\overline{\varepsilon}\, b \wedge \neg(c \,\overline{\varepsilon}\, a))$ is $SAT^{F,u}_{OBJ}$, there exists a valuation A^s differing from A^u at most in the value assigned to c, such that $c \,\overline{\varepsilon}\, b \wedge \neg(c \,\overline{\varepsilon}\, a)$ is $SAT^{F,s}_{OBJ}$. It implies that $c \,\overline{\varepsilon}\, b$ is $SAT^{F,s}_{OBJ}$. But it cannot be satisfied. Hence, by contradiction, we obtain the left side of the equivalence. Q.E.D.

4.8.2 Axiom 6

Sobociński's axiomatization has $<$ as the only primitive functor. Let us follow the Definition 37 according to the definition of satisfaction$^\Phi$. Our semantics yields the result that interpretation$^\Phi$ of $<$ is the same as the interpretation$^\Phi$ of ε .

Theorem 14 *Axiom 6 is valid$^\Phi$.*

PROOF: *Implication to the right*: Assume $a < b$ is $SAT^{\Phi,u}_{OBJ}$. Hence $V^u(a) = V^u(b) = \Phi$. There exists a valuation A^s differing from A^u at most in the value assigned to c, such that $V^s(c) = \Phi$. Obviously $c < a$ is $SAT^{\Phi,s}_{OBJ}$. Therefore, $\exists_c c < a$ is $SAT^{\Phi,u}_{OBJ}$.

Now assume additionally, that $c < a$ is $SAT^{\Phi,u}_{OBJ}$. Then $V^u(c) = \Phi$. Since $V^u(b) = \Phi$, it is clear that $V^u(c) = V^u(b) = \Phi$, and $c < b$ is $SAT^{\Phi,u}_{OBJ}$.

Therefore $\forall_c(c < a \rightarrow c < b)$ is $SAT^{\Phi,u}_{OBJ}$. Then also $\forall_c(c < a \wedge \forall_d(d < c \rightarrow c < d) \rightarrow c < b)$ is $SAT^{\Phi,u}_{OBJ}$.

Implication to the left: Assume

$$\exists_c c < a \wedge \forall_c(c < a \wedge \forall_d(d < c \rightarrow c < d) \rightarrow c < b)$$

is $SAT^{\Phi,u}_{OBJ}$.

Since $\exists_c c < a$ is $SAT^{\Phi,u}_{OBJ}$, there exists a valuation A^s differing from A^u at most in the value assigned to c, such that $V^s(c) = V^s(a) = \Phi$. Since $V^s(a) = V^u(a)$, clearly $V^u(a) = \Phi$.

According to our assumption, $\forall_c(c < a \wedge \forall_d(d < c \rightarrow c < d) \rightarrow c < b)$ is $SAT^{\Phi,u}_{OBJ}$. Moreover, $\forall_d(d < c \rightarrow c < d)$ is valid$^{\Phi}$, since identity for sets is symmetrical. Hence $\forall_c(c < a \rightarrow c < b)$ is $SAT^{\Phi,u}_{OBJ}$.

Obviously $a < a$ is $SAT^{\Phi,u}_{OBJ}$(because $V^u(a) = V^u(a) = \Phi$). Then, in accordance with the valid implication above ($\forall_c(c < a \rightarrow c < b)$), also $a < b$ is $SAT^{\Phi,u}_{OBJ}$. This result ends our proof. Q.E.D.

We can also follow this definition by means of the definition of satisfaction^F. The resulting interpretation^F of $<$ is the same as the interpretation^F of ε .

Theorem 15 *Axiom 6 is valid^F.*

PROOF: *Implication to the right*: Assume $a < b$ is $SAT^{F,u}_{OBJ}$. Since it cannot be satisfied, we have contradiction, and the right side of the equivalence follows obviously.

Implication to the left: Assume $\exists_c c < a$ is $SAT^{F,u}_{OBJ}$. Then for some valuation A^s differing at most in the value assigned to c, $c < a$ is $SAT^{F,s}_{OBJ}$. Since it cannot be the case, we obtain contradiction, which yields the needed statement. Q.E.D.

4.8.3 Axiom 7

The primitive term of Ontology based on this axiom is $\subset$. Let us find a non-standard interpretation of this functor, following the definition of satisfaction$^{\Phi}$ and the Definition 39.

It is defined by means of $\forall_c(c\,\varepsilon\, a \rightarrow c\,\varepsilon\, b)$. Hence, we can introduce the following:

Definition 47 $\ulcorner \mu_k \subset \mu_i \urcorner$ *is satisfied*$^\Phi$ *in* OBJ *by a valuation* A^u if and only if

$$A_k = \Phi \to A_i = \Phi$$

It is clear that this interpretation is far from standard. We shall now prove the following:

Theorem 16 *Axiom 7 is valid*$^\Phi$*.*

PROOF: Implication to the right: Assume $a \subset b$ is $SAT^{\Phi,u}_{OBJ}$. It implies that if $V^u(a) = \Phi$ then also $V^u(b) = \Phi$.

Assume additionally that $c \subset a$ is $SAT^{\Phi,u}_{OBJ}$and $V^u(c) = \Phi$. It means, that if $V^u(c) = \Phi$, then also $V^u(a) = \Phi$. From this assumptions we obtain that $V^u(a) = \Phi$, and, since $V^u(a) = \Phi \to V^u(b) = \Phi$, obviously $V^u(b) = \Phi$.

Therefore, also $c \subset b$ is $SAT^{\Phi,u}_{OBJ}$.

Implication to the left: This part of the proof will be indirect. Assume the following:

1. $a \subset b$ is not $SAT^{\Phi,u}_{OBJ}$.
2. $\forall_c (c \subset a \wedge \forall_{d,e}(d \subset c \to c \subset d \vee d \subset e) \to c \subset b)$ is $SAT^{\Phi,u}_{OBJ}$.

Obviously (by the semantic correlate of universal instantiation) also $a \subset a \wedge \forall_{d,e}(d \subset a \to a \subset d \vee d \subset e) \to a \subset b)$ is $SAT^{\Phi,u}_{OBJ}$.

It is clear that $V^u(a) = \Phi \to V^u(a) = \Phi$. Hence, $a \subset a$ is $SAT^{\Phi,u}_{OBJ}$.

Now, from the first assumption, it is not the case that $V^u(a) = \Phi \to V^u(b) = \Phi$. Hence $V^u(a) = \Phi \wedge V^u(b) \neq \Phi$ is $SAT^{\Phi,u}_{OBJ}$.

Now assume that $d \subset a$ is $SAT^{\Phi,u}_{OBJ}$. Clearly, no matter what the value of d is, either $V^u(d) = \Phi$, or $V^u(d) \neq \Phi$. In the first case, $V^u(a) = \Phi \to V^u(d) = \Phi$, and $a \subset d$ is $SAT^{\Phi,u}_{OBJ}$. In the second case, obviously $V^u(d) = \Phi \to V^u(e) = \Phi$ (no matter, what the value of e is, since the antecedent is false). In both cases, $a \subset d \vee d \subset e$ is $SAT^{\Phi,u}_{OBJ}$.

Therefore, $\forall_{d,e}(d \subset a \to a \subset d \vee d \subset e)$ is $SAT^{\Phi,u}_{OBJ}$.

Now, by *modus ponendo ponens*, we obtain the consequence that $a \subset b$ is $SAT^{\Phi,u}_{OBJ}$. This gives us contradiction with the assumption of indirect proof. This result ends our proof. Q.E.D.

Clearly, our method works also when we follow the definition of satisfaction^F. According to the Definition 39 we obtain:

Definition 48 *$\lceil \mu_k \subset \mu_i \rceil$ is satisfied^F in OBJ by a valuation A^u* if and only if $A_k = A_k$.

In this interpretation $\subset$ is a *verum* functor which yields the value 1 for any arguments. We claim:

Theorem 17 *Axiom 7 is valid^F.*

PROOF: *Implication to the right*: Assume $a \subset b$ is $SAT^{F,u}_{OBJ}$.

Clearly, $c \subset b$ is $SAT^{F,u}_{OBJ}$, since $V^u(c) = V^u(c)$. Therefore, also the whole implication on the right side of equivalence is $SAT^{F,u}_{OBJ}$.

Implication to the left: Similarly — $a \subset b$ is $SAT^{F,u}_{OBJ}$(because $V^u(a) = V^u(a)$). Therefore whole the implication which has $a \subset b$ as consequent is $SAT^{F,u}_{OBJ}$. This ends our proof. Q.E.D.

4.9 Some Other Non-Standard Interpretations

As we have said in (Urbaniak [47]), there are 2^{16} possible interpretations of a given functor of the category $\frac{s}{n,n}$. The claim that an axiom determines exactly one of this meanings is therefore quite a strong claim. In this section we shall indicate that the two interpretations presented above are not the only two possible non-standard interpretation.

However, our treatment of this issue will be less precise that the deliberations led to this point.

We will only prove our theorems for the Axiom 4. Clearly, those claims can be applied to other axioms. We omit these theorems and their proofs just for the sake of convenience.

We can, for example, introduce the following inductive condition of satisfaction.:

Definition 49 *$\mu_k \varepsilon \mu_i$ is satisfied$^\alpha$ in OBJ by A^u iff $\exists_{x \neq y}[x \in A_k \wedge y \in A_k \wedge A_k = A_i]$.*

In other words, in interpretation$^\alpha$ we understand ε as identity of extensions having more that one element.

Theorem 18 *Axiom 4 is valid*$^{\alpha}$.

PROOF: *Implication to the right*: Assume $a\ \varepsilon\ b$ is $SAT^{\alpha,u}_{OBJ}$. It means that $V^u(a)$ and $V^u(b)$ are non empty, identical and not unary. There is a valuation A^s which differs from A^u at most in the fact that $Vs(c) = V^s(a)$. Clearly $a\ \varepsilon\ c \wedge c\ \varepsilon\ b$ is $SAT^{\alpha,s}_{OBJ}$.

Therefore, $\exists_c(a\ \varepsilon\ c \wedge c\ \varepsilon\ b)$ is $SAT^{\alpha,u}_{OBJ}$.

Implication to the left: Assume that $\exists_c(a\ \varepsilon\ c \wedge c\ \varepsilon\ b)$ is $SAT^{\alpha,u}_{OBJ}$. Then, there is a valuation A^s which differs from A^u at most in the fact that $Vs(c) = V^s(a)$. Clearly $a\ \varepsilon\ c \wedge c\ \varepsilon\ b$ is $SAT^{\alpha,s}_{OBJ}$ and sets $V^s(a), V^s(b), V^s(c)$ are identical, nonempty and not unary.

Since $V^s(a) = V^u(a)$ and $V^s(b) = V^u(b)$, it is obvious that $V^u(a)$ and $V^u(b)$ are identical, nonempty and not unary. Therefore, $a\ \varepsilon\ b$ is $SAT^{\alpha,u}_{OBJ}$. This ends the proof. Q.E.D.

Definition 50 $\mu_k\ \varepsilon\ \mu_i$ *is satisfied*$^{\beta}$ *in* OBJ *by* A^u *iff* $\exists!_x[x \in A_k \wedge A_k = A_i]$.

According to the definition of satisfaction$^{\beta}$, we interpret ε as the identity of unary extensions of given arguments.

Theorem 19 *Axiom 4 is valid*$^{\beta}$.

Proof of Theorem 19 is very similar to the proof of Theorem 18.

Definition 51 $\mu_k\ \varepsilon\ \mu_i$ *is satisfied*$^{\gamma}$ *in* OBJ *by* A^u *iff* $\exists_x[x \in A_k \wedge A_k = A_i]$.

According to the definition of satisfaction$^{\gamma}$, we interpret ε as identity of nonempty extensions of given arguments.

Theorem 20 *Axiom 4 is valid*$^{\gamma}$.

As before, the proof of Theorem 20 is very similar to the proof of Theorem 18.

The list of hitherto given interpretations of primitive functors of Leśniewski's Ontology is not a complete list of possible interpretations of primitive constants of Ontology which keep axioms valid.

4.10 A General Case — Numerical Identity

Some of the interpretations of ε mentioned above represent a particular instances of a more general situation. These are, to speak freely, these interpretations of ε , which imply identity.[55]

In a language with identity of individuals we can easily define predicates: 'has exactly (at least, at most) n elements'. Hence, we can introduce $\aleph_0$ definitions of satisfaction by constructing the basic condition according to the scheme:

> [IDENTITY SCHEME] $\mu_k \; \varepsilon \; \mu_i$ is satisfied$^{n(\geq n/\leq n)}$ in OBJ by A^u
> iff $A_k = A_i$ and A_k has exactly (at least/ at most) n elements.

where n is a natural number.

We observe:

Theorem 21 *Any satisfaction defined in accordance with the IDENTITY SCHEME makes all the enumerated axioms of Ontology valid.*

We have already given proofs of this theorem for some particular notions of numerical satisfaction — satisfaction$^{\Phi}$, satisfaction$^{\alpha}$, satisfaction$^{\beta}$, satisfaction$^{\gamma}$. Proof of this theorem is an easy generalization of proofs of corresponding claims for these interpretations. Such proofs have been already given. The proof needed now is constructed from those by replacing a particular numerical description[56] by 'has(ve) exactly (at least/ at most) n elements'.

4.11 Rule of Extensionality

We shall begin with the application of this rule to the basic language, i.e. with the schema

$$\text{From } \forall_x[x \; \varepsilon \; \lambda \equiv x \; \varepsilon \; \delta] \text{ infer } \forall_\chi[\chi(\lambda) \equiv \chi(\delta)]$$

It will be convenient to begin with a lemma saying that if the rule works for $\chi = \varphi$ and for $\chi = \psi$, then it works for expressions composed from φ and ψ by the rules of BL (conjunction, negation, quantification). This is the most trivial part of a full proof of our main claim for this rule. We can prove it in general,

[55] Strictly speaking: a given interpretation ψ of atomic sentences defined by the condition of the kind: '$a \; \varepsilon \; b$ is $SAT^{\psi,u}_{OBJ}$ iff $\psi(a,b)$' implies identity iff $\psi(a,b)$ implies $V^u(a) = V^u(b)$.

[56] Like, for example, 'is non-empty' or 'has at least two elements'.

and later on prove only the first condition of inductive proof for different cases separately.

Lemma 10 *If the two schemata are validity-preserving:*

$$\textit{From } \forall_x[x\ \varepsilon\ \lambda \equiv x\ \varepsilon\ \delta] \textit{ infer } [\varphi(\lambda) \equiv \varphi(\delta)]$$

$$\textit{From } \forall_x[x\ \varepsilon\ \lambda \equiv x\ \varepsilon\ \delta] \textit{ infer } [\psi(\lambda) \equiv \psi(\delta)]$$

then, also the following schemata are validity-preserving:

1. *From* $\forall_x[x\ \varepsilon\ \lambda \equiv x\ \varepsilon\ \delta]$ *infer* $[\varphi(\lambda) \wedge \psi(\lambda) \equiv \varphi(\delta) \wedge \psi(\delta)]$
2. *From* $\forall_x[x\ \varepsilon\ \lambda \equiv x\ \varepsilon\ \delta]$ *infer* $[\neg\varphi(\lambda) \equiv \neg\varphi(\delta)]$
3. *From* $\forall_x[x\ \varepsilon\ \lambda \equiv x\ \varepsilon\ \delta]$ *infer* $[\forall_\mu\varphi(\lambda) \equiv \forall_\mu\varphi(\delta)]$

Proof: Assume that the rule is validity-preserving for φ and ψ. Assume that $\forall_x[x\ \varepsilon\ \lambda \equiv x\ \varepsilon\ \delta]$ is valid. Hence, we can infer $\varphi(\lambda) \equiv \varphi(\delta)$ and $\psi(\lambda) \equiv \psi(\delta)$. The cases of conjunction and negation result from, respectively:

$$(p \equiv q) \wedge (r \equiv s) \to ((p \wedge r) \equiv (q \wedge s))$$

$$(p \equiv q) \to (\neg p \equiv \neg q)$$

Regarding quantification: Since $\forall_\mu(\varphi(\lambda))$ and $\forall_\mu(\varphi(\delta))$ must have λ and δ as free variables[57], clearly $\lambda \neq \mu$ and $\delta \neq \mu$. We have now two possibilities: either μ occurs in φ, or not. If it does not occur in φ, it, to say it loosely, it does not contribute to the value of $\varphi(\lambda)$ or $\varphi(\delta)$. If μ occurs in ϕ as a bound variable, the addition of superfluous quantification also changes nothing in the value of whole expressions. So the only interesting case remaining is that μ occurs in ϕ as a free variable.

Note that we already have $\varphi(\lambda) \equiv \varphi(\delta)$.

If this is valid, then for any OBJ and for any valuation of free variables (μ among them) $\varphi(\lambda)$ is equivalent to $\varphi(\delta)$. Clearly, then, [for any valuation of μ, $\varphi(\lambda)$] is equivalent to [for any valuation of μ, $\varphi(\delta)$]. Hence, $[\forall_\mu\varphi(\lambda) \equiv \forall_\mu\varphi(\delta)]$ is valid.

Q.E.D.

[57] See the formulation of the rule of extensionality.

With the aid of this result, we can easily give proofs that the rule of extensionality is validity-preserving for a given interpretation. Since lemma 10 works for any interpretation, to complete an inductive proof it will suffice to prove a claim about the basic condition only.

Theorem 22 *Rule of extensionality is validity$^{\Phi}$-preserving.*

PROOF: To prove this claim we shall prove that if $\forall_x[x \varepsilon \lambda \equiv x \varepsilon \delta]$ is valid$^{\Phi}$, then the following schemata are valid$^{\Phi}$:

1. $\gamma \varepsilon \lambda \equiv \gamma \varepsilon \delta$
2. $\lambda \varepsilon \gamma \equiv \delta \varepsilon \gamma$

In fact, the first case is obvious (check our semantics for the universal quantifier). The second results from the fact that the relation of 'being simultaneously empty' is symmetrical.

Indeed, these two cases exhaust all possibilities of 'atomic' formulas in which both λ and δ occur at the same position. Together with lemma 10 this fact yields a complete inductive proof for BL.

Q.E.D.

We can prove a more general claim:

Theorem 23 *Rule of extensionality preserves validity in BL for any interpretation of ε constructed in accordance with the schema of numerical identity.*

PROOF: Assume that

$$\forall_x(x \varepsilon \lambda \equiv x \varepsilon \delta)$$

is valid. It follows that for any OBJ and any valuation u of name variables in this model, any (and only) subset of OBJ which is nI with $EV^u(\lambda)$ is nI with $EV^u(\delta)$.[58] This implies that $EV^u(\lambda)$ and $EV^u(\delta)$ are nI with each other (and each of them is nI with itself). If these semantic values are identical, it does not matter which of these: λ or δ, occur in an expression — since satisfaction is defined in terms of extensions of names and name

[58] We say shortly 'nI' instead of 'n-numerically identical' where 'n-numerical identity' replaces 'being identical and having exactly/at least/at most n elements'. We hope that this abbreviation will not cause any ambiguity.

variables and their power (the latter is preserved by identity). So, in fact, always any expression with λ is equivalent with the same expression after replacing λ by δ. Q.E.D.

This proof seems to be quite plausible, *mutatis mutandis*, for the case of name- and sentence-formative functors. Hence:

Hypothesis 3 *Rule of extensionality preserves validity in the full language of Ontology for any interpretation of* ε *constructed in accordance with the schema of numerical identity.*

Almost all interpretations considered in this paper fall under the schema for numerical identity. The only interpretation which need an independent proof is the notion of satisfaction^F.

Theorem 24 *Rule of extensionality preserves validity^F in BL.*

Proof: We need to prove only the basic case (for atomic expressions). Indeed, the antecedent of the rule of extensionality itself is a valid expression, for any δ and λ (both arguments of the equivalence are always false).[59] Now, clearly all atomic sentences are equivalent — they are always false. Thanks to lemma 1 we obtain a full inductive proof. Q.E.D.

Presumably, the above proof works, *mutatis mutandis*, for the full language of Ontology.

4.12 Rule of Definition

It is quite clear that:

Theorem 25 *The rule of definition is validity^F-preserving in the basic language of Ontology.*

Proof: For the case of definitions proposition-formative functors, the case is simple, because the interpretation of ε is not relevant for the validity of such definitions. For the case of definitions of nominal constants and of name-formative functors note, that such a definition:

[59]It does not indicate, however, that we can (syntactically) prove this antecedent in the axiomatic system under consideration.

(a) Has an atomic expression as definiendum.

(b) Has at least one atomic expression as a part of a conjunction in the definiens.

Since any atomic expression is unsatisfiable^F, it means that both definiendum and definiens of any such definition are unsatisfiable. Hence, any such definition (since it is an equivalence) is valid. Q.E.D.

Nevertheless, an anonymous referee of this section[60] has given an interesting argument for the claim that in the interpretation$^\Phi$ of ε the rule of definition 'is not truth preserving' in the Φ-interpretation. This arguments sheds some light on important issues connected with this rule. I am really grateful to this referee, because his comment, relevant indeed, reminded me of some issues neglected in the first version of this paper.

Let us start with the argument itself before adding a comment.[61]

Argument

The definition of the empty name is:

$$\mathrm{D}\bigcap: \qquad \forall_a (a \;\varepsilon\; \bigcap \equiv a \;\varepsilon\; a \land \neg a \;\varepsilon\; a)$$

When we introduce this definition, we can prove in Ontology:

$$\neg \bigcap \;\varepsilon\; \bigcap$$

which is false according to Φ-interpretation.

Comment In fact, the rule of definition allows us to introduce $\mathrm{D}\bigcap$ as a theorem. Since it is only 'one-step'-rule, its validity-preserving property consists in the fact that any expression introduced by means of this rule is valid.[62]

[60] The section on non-standard interpretations has been accepted for publication in the Australasian Journal of Logic (edited by Greg Restall). I mean the referee of this Journal.

[61] I wholeheartedly hope that the referee does not mind quoting his argument.

[62] Please note, that for our purpose it suffices to consider the property of being validity-preserving (which in fact, for rules of introducing sentences, coincides with being truth-preserving). As we will see, expressions introduced by means of this rule will not have to be sentences. Especially, the so-called quasi-constants (we shall explain this notion soon) are, in some sense, free in such 'definitions' in some non-standard interpretations. To resume — validity preserving is the weakest property we need, so we will be concerned with it (and not with truth-preserving property) in our further deliberations.

On the other hand, it is a wholly different question, whether expressions introduced by means of this rule determine constant denotations of expressions defined in these definitions.

This may be clearly seen on the example of D$\bigcap$. Indeed, we can see that it does not 'determine' the standard denotation of $\bigcap$. Colloquially speaking, we may read D$\bigcap$ in our Φ-interpretation rather as:[63]

> For any name a, a is empty together with $\bigcap$ iff (a is empty with itself and it is not the case that a is empty with itself).

It follows that for no a it is the case that $a \varepsilon \bigcap$. But, in our interpretation (Φ) it does not mean that $\bigcap$ is the empty name. To the contrary, it means that no name is empty together with $\bigcap$, so we should rather call $\bigcap$ 'the non-empty name' (whether the usage of 'the' before 'non-empty name' is legitimate is itself debatable).

This fact indicates that when we apply the Φ-interpretation (and possibly, some other interpretations) to definitions of Ontology, they may not determine the semantics of defined expressions.[64] We can now understand $\bigcap$ in many different ways, as far as we assign in any OBJ a non-empty set as its semantic correlate.

Anyway, if we interpret $\bigcap$ as a non-empty name (as we are forced to do by D$\bigcap$ in the Φ-interpretation), no surprise (and, moreover, no contradiction) that we obtain as a result:

$$\neg \bigcap \varepsilon \bigcap$$

But we *cannot* read it as 'the empty name is not empty together with itself' (which is false), but rather: 'a non-empty name is not empty together with itself' (which is true).

To summarize, when we change the meaning of constants of a given language, this shapes also our account of definitions built in this language — we have to apply our new semantics also to definitions.

[63] I hope that this reading will suffice. Anyone who feels that the full formal reading is necessary, is free to read it accordingly.

[64] It is interesting question, whether definitions originally in fact determined such semantics. Anyway, even if they do not determine this semantics, it does not mean that they are not valid.

General consideration Nevertheless, the referee is quite right: the rule is not validity-preserving in some non-standard interpretations. It surely is not validity-preserving in any numerical interpretation where $n \geq 2$, if we put no restrictions on the validity/unsatisfiability of χ occurring in the definition. Consider the following:

$$D* \quad \forall_a [a \varepsilon * \equiv a \varepsilon a \wedge (a \varepsilon a \vee \neg a \varepsilon a)]$$

This expression obviously fails in numerous cases of n-identity. For example, read ε as denoting the relation: 'has at most 5 elements and is identical with', and let χ be valid (like in $D*$). Let OBJ be a model under consideration. We obtain the result that $*$ denotes a subset $V(*)$ of OBJ, such that any $A \subseteq OBJ$ which has at most n elements, (has at most n elements and) is identical with $V(*)$. Let $OBJ = \{x, y, z\}$, $B = \{1, 2\}$, $C = \{2, 3\}$, $B, C \subseteq OBJ$. Clearly both B and C have at most 5 elements. Hence, $A = V(*)$, $B = V(*)$. Therefore: $A = B$, which is false.

There are, however, at least two non-standard interpretations, in which this rule is validity preserving - F-interpretation, and Φ-interpretation.

Generally, the question whether the rule of definition is validity preserving in non-standard interpretations gives rise to quite a complicated deliberations. The restrictions put on possible non-standard interpretation of ε by introducing the rule of definition (or the other way round - the restrictions put on this rule by introducing some non-standard interpretations) constitute a wide problem which I cannot discuss fully in this paper. I hope that I will be able to come back to this issue in a separate paper.

5 Leśniewskian Approach to Russell's Paradox

5.1 Paradoxes

We shall begin with listing Leśniewskian formulations of paradoxes connected with the notion of class, as already formalized by Sobociński (Sobociński, [43]). We will only give assumptions and final conclusions, the full proofs can be found in Sobociński's paper. We shall use the language of Ontology, giving all readings in an intuitive manner, for the sake of simplicity avoiding any formal semantics of Ontology.[65]

The notion of an element is related to the notion of class in accordance with the following:

$$\text{EC} \qquad \forall_{a,b}[b \,\varepsilon\, el(a) \equiv \exists_c (a \,\varepsilon\, Kl(c) \wedge b \,\varepsilon\, c)]$$

where $el(a)$ is a general name of elements of a, and $Kl(a)$ is the class of objects named by the name a.

5.1.1 First Paradox

consists in deducing a contradiction from the following two assumptions:[66]

$$\text{A1} \qquad \forall_a \exists_b b \,\varepsilon\, Kl(a)$$

$$\text{A2} \qquad \forall_{a,b,c,d}[a \,\varepsilon\, Kl(c) \wedge a \,\varepsilon\, Kl(d) \wedge b \,\varepsilon\, d \rightarrow b \,\varepsilon\, c]$$

A1 claims that for each name, there is a class of objects named by this name. A2 claims that if there is a unique object named by a, which simultaneously is both the class of objects named by c and the class of objects named by d, then any single object named by d is also named by c. Or, in other words, if the class of two names is the same, whatever is named by one of them is named by the other.

We introduce the definition:

$$\text{D1} \qquad \forall_a[a \,\varepsilon\, \star \equiv (a \,\varepsilon\, a \wedge \forall_b(a \,\varepsilon\, kl(b) \rightarrow \neg\, a \,\varepsilon\, b))]$$

which means that an object is a $\star$ iff it exists and it is not named by any name of which it is a class.[67]. Now, we can deduce the following:

[65] We modify the original notation when it is convenient.

[66] In naming formulas which occur in Sobociński's paper, we mainly follow (Sobociński, [43]).

[67] Sobociński reads the definition differently:

$$\text{A6} \qquad \forall_A \neg(A \,\varepsilon\, Kl(\star))$$

$$\text{A7} \qquad \exists_A (A \,\varepsilon\, Kl(\star))$$

which contradict each other.

5.1.2 Second Paradox

is obtained from the first one by replacing A2 by the two following expressions:

$$\text{B1} \qquad \forall_{a,b,c}(a \,\varepsilon\, Kl(c) \wedge b \,\varepsilon\, c \rightarrow b \,\varepsilon\, el(a))$$

$$\text{B2} \qquad \forall_{a,b,c}(a \,\varepsilon\, Kl(c) \wedge b \,\varepsilon\, el(a) \rightarrow b \,\varepsilon\, c)$$

since A2 is derivable from $\{B1, B2\}$.[68]

5.1.3 Third Paradox

occurs, if we accept A2 (or, alternatively, B' or $\{B1, B2\}$), but instead of A1 accept:

$$\text{C1} \qquad \forall_{b,c}(b \,\varepsilon\, c \rightarrow \exists_a (a \,\varepsilon\, Kl(c)))$$

which says that for each non-empty name c there is a class of c.

With an additional assumption:

$$\text{C2} \qquad \exists_{a,b}(a \,\varepsilon\, a \wedge b \,\varepsilon\, b \wedge \neg(a = b))$$

> According to this definition we can say that "$A \,\varepsilon\, \star$" means that "A is a set (class) which is not an element of itself" or in other words "A is a class which is not subordinated to itself". (Sobociński, [43, p. 16])

In fact, it is a little bit too strong reading of the *definiens*. The definiens does not claim that A is a class at all — it only says that *if* it is a class of a name a, *then* the object named by A is not named by a.

[68] Clearly, instead of $\{B1, B2\}$ we could use:

$$\text{B'} \qquad \forall_{a,b,c}(a \,\varepsilon\, Kl(c) \rightarrow (b \,\varepsilon\, c \equiv b \,\varepsilon\, el(a)))$$

.

stating that there are at least two different objects,[69] and by means of D1 we obtain the conclusion:

$$\text{C10} \qquad \forall_{a,b}(a \,\varepsilon\, a \wedge b \,\varepsilon\, b \rightarrow a = b)$$

which contradicts C2.

5.1.4 Fourth Paradox

takes place, if we try to replace A2 by a weaker claim, coming from Frege's proposal of correction of his system for the foundations of mathematics. The proposal is to be found in (Frege, [13, pp. 262-265]). In the language of Ontology it has the form:

$$\text{E1} \qquad \forall_{a,b,c,d}(a \,\varepsilon\, Kl(c) \wedge a \,\varepsilon\, Kl(d) \wedge b \,\varepsilon\, d \wedge \neg(b \,\varepsilon\, Kl(d)) \rightarrow b \,\varepsilon\, c)$$

and means what follows. If there is a unique object which simultaneously is both: the class of c and the class of d; and the object named by b is not the class of d and is named by d, then the object named by b is named by c.

With two other claims:

$$\text{E2} \qquad \forall_{a,b,c}(a \,\varepsilon\, Kl(c) \wedge b \,\varepsilon\, Kl(c) \rightarrow a = b)$$

$$\text{E3} \qquad \exists_{a,b,c}(a \,\varepsilon\, a \wedge b \,\varepsilon\, b \wedge c \,\varepsilon\, c \wedge \neg(a = b) \wedge \neg(a = c) \wedge \neg(b = c))$$

we obtain a contradiction. E2 postulates that for any name there is at most one class of this name, and E3 — that there are at lest three different objects. The assumption of $\{C1, E1, E2, E3\}$ with the aid of the definition:

$$\text{D3} \qquad \forall_a(a \,\varepsilon\, \clubsuit \equiv a \,\varepsilon\, a \wedge \forall_b(a \,\varepsilon\, Kl(Kl(b)) \rightarrow \neg(a \,\varepsilon\, c)))$$

which repeats D1, but 'on a higher level', leads to:

$$\text{E17} \qquad \forall_{a,b}(a \,\varepsilon\, a \wedge b \,\varepsilon\, b \rightarrow a = b)$$

which states that there is at most one object. This obviously contradicts E3.

[69] The expression $a = b$ is equivalent to $a \,\varepsilon\, b \wedge b \,\varepsilon\, a$.

5.1.5 Fifth Paradox

is the Ontological correlate of the 'usual formulation' of Russell's Paradox. We assume

$$\text{L1} \qquad \forall_a (a \; \varepsilon \; \Lambda \equiv \neg(a \; \varepsilon \; a))$$

and obtain:[70]

$$\text{L2} \qquad \Lambda \; \varepsilon \; \Lambda \equiv \neg(\Lambda \; \varepsilon \; \Lambda)$$

5.2 Leśniewski's Solution

Leśniewski' s short solution of the *fifth paradox* differs from his solution for other paradoxes ad gives rise to no controversy, therefore we shall present it shortly and elaborate on his solution for the rest of difficulties.

5.2.1 Solution of the fifth paradox

The answer for the *fifth paradox* is quite simple. L1 does not satisfy requirements put on definitions in Leśniewski's Ontology.

The schema for defining a nominal constant, say α, is as follows:

$$b \; \varepsilon \; \alpha \equiv b \; \varepsilon \; b \wedge \chi$$

where χ is a propositional expression of the language of Ontology. Therefore, L1 should rather have the form:

$$\text{D5} \qquad \forall_a (a \; \varepsilon \; \Lambda \equiv a \; \varepsilon \; a \wedge \neg(a \; \varepsilon \; a))$$

From this definition we obtain:

$$\Phi \qquad \forall_a \neg(a \; \varepsilon \; \Lambda)$$

which shows that D5 is simply an alternative definition of the empty set.

[70]Strictly speaking, it is not a translation of Russell's Paradox into the language of Ontology, because ' ε ' has a different meaning than '$\in$' has.

5.2.2 Solution of first four paradoxes

As Sobociński remarks:

> The fact of the existence of the paradox . . . forced Leśniewski to submit the terms which appear in the presuppositions to a fundamental analysis. Since he was sure of the laws of logic on which our reasoning rests as well as the manner of understanding the logical constants contained in these laws, he analyzed the terms "class" and "element" which appear in the presuppositions. The examination of this problem led him to the conclusion that a superficial determination of these terms . . . is insufficient and mistaken, and, in fact, provokes the appearance of the contradiction. (Sobociński, [43, p. 30]).

Therefore, Leśniewski made a distinction between two notions of class: the distributive notion and the collective one.

5.2.3 Distributive class and paradoxes

Here is a very important description of how Leśniewski[71] understood the notion of class in the distributive sense:

> The expression "class(a)" in the distributive sense is nothing more than a fictitious name which replaces the well-known term of classical logic, "the extension of the objects a". If one takes the term in this sense, the formula "$A\ \varepsilon\ Kl(a)$" means the same thing as "A is an element of the extension of the objects a", or, more briefly, "A is a". In which case, the formula "Socrates is Kl(white)" means the same thing as "Socrates is white". Thus, the understanding of "class" in the distributive sense would reduce the formula "$A\ \varepsilon\ Kl(a)$" to the purely logical formula "$A\ \varepsilon\ a$" where " ε " is understood as the connective of the individual proposition.(Sobociński, [43, p. 31])

Hence we obtain the distributive interpretation of the class:

$$\text{DISTR} \qquad \forall_{a,b}(a\ \varepsilon\ Kl(b) \equiv a\ \varepsilon\ a \wedge a\ \varepsilon\ b)$$

since $a\ \varepsilon\ b$ implies $a\ \varepsilon\ a$ we can simply say:

[71] According to Sobociński.

$$\text{DISTR'} \qquad \forall_{a,b}(a \,\varepsilon\, Kl(b) \equiv a \,\varepsilon\, b)$$

With the use of DISTR' we can remove the use of 'Kl' in the presuppositions which led to contradictions. Simply, for any a, b, instead of $a \,\varepsilon\, Kl(b)$ write $a \,\varepsilon\, b$. In this interpretation some of presuppositions become false: A1, A2, B2, C2, E2. Since there is no paradox in obtaining odd consequences from false presuppositions, our problems disappear.

5.2.4 Collective class and paradoxes

In the collective sense, the class of object named by a shared[72] name, say a, is a real new object - just as a heap of stones is not a stone, but is a collective class of stones in that heap. Moreover, there are not only stones among the collective elements of that heap — each part of it is its element.

In other words, Leśniewski originally used the word *ingredjens* (Leśniewski, [29, p. 264])[73] in the following manner: each object is an ingredient of itself and each proper part of this object is an ingredient of this object, and there are no other ingredients of this object than itself or its proper parts.

By a collective class of a name a we mean the object composed of all designates of a. Any ingredient of such formed composite object is an element of such a class.[74]

In this interpretation of the notion of class the following presuppositions are false: A1, A2, B2, E1. This fact shows that if we understand class in the collective sense, the paradoxes mentioned above disappear.

5.3 Doubts

5.3.1 The distributive notion of class

First of all, let us emphasize the standard semantics of ε in Leśniewski's Ontology. For two names, a and b, the sentence $a \,\varepsilon\, b$ is true iff a names exactly one object, and this object is named by b.[75]

[72] I.e. naming more than one object. The term 'shared' in application to Ontology comes from (Lejewski, [24]).

[73] We will use 'ingredient'.

[74] For a detailed account of Leśniewski's notion of class see (Pietruszczak, [36, p. 1-38]).

[75] This idea can be easily translated into formal semantics. We have done it in (Urbaniak [47]).

Now, let us get back to the quotation from Sobociński:

> The expression "class(a)" in the distributive sense is nothing more than a fictitious name which replaces the well-known term of classical logic, "the extension of the objects a". If one takes the term in this sense, the formula "$A \varepsilon Kl(a)$" means the same thing as "A is an element of the extension of the objects a", or, more briefly, "A is a". In which case, the formula "Socrates is Kl(white)" means the same thing as "Socrates is white". Thus, the understanding of "class" in the distributive sense would reduce the formula "$A \varepsilon Kl(a)$" to the purely logical formula "$A \varepsilon a$" where "ε" is understood as the connective of the individual proposition.(Sobociński, [43, p. 31])

He seems to be claiming:

SOB1 $\quad Kl(a) =$ the extension of the objects a

If we instead of 'the extension of the objects a' shall write '$ext(a)$', then his second claim has the form:

SOB2 If SOB1, then $(A \varepsilon Kl(a) \equiv A \varepsilon$ an element of $ext(a))$

Given our semantics for ε, SOB2 is clearly false. Our identity in SOB1 would allow us only to introduce:

SOB2' If SOB1, then $(A \varepsilon Kl(a) \equiv A \varepsilon\ ext(a))$

To obtain SOB2 from SOB1 we would need also a presupposition that:

SOB3 $\quad ext(a) =$ an element of $ext(a)$

or at least some rule or theorem which would allow us to replace '$ext(a)$' by 'an element of $ext(a)$'. Nevertheless, if we understand 'extension' in the distributive manner, it is really far from obvious that SOB3 or such a rule are acceptable at all. Otherwise, we would have a theorem:

$$\text{SOB4} \qquad \forall_a (ext(a) \varepsilon\ ext(a) \to ext(a) \varepsilon \text{ an element of } ext(a))$$

which, together with SOB1 yields:

$$\text{SOB5} \qquad \forall_a (Kl(a) \varepsilon\ Kl(a) \to Kl(a) \varepsilon \text{ an element of } Kl(a))$$

This would mean that any existing *distributive* class is an element of itself. Considering the fact that the relation of belonging to a distributive class is not taken to be reflexive, it is a really odd result.

Even if we accept DISTR', no matter what the argumentation for this claim may be, still we have not removed paradoxes. It is right that in the interpretation expressed by DISTR' four paradoxes disappear. Nevertheless, we can construct a new ones.

In Ontology we can accept the definition of a sum of names:

$$\text{D}\cup \qquad \forall_{a,b,c}(a\ \varepsilon\ b \cup c \equiv a\ \varepsilon\ b \vee a\ \varepsilon\ c)$$

so for each a, b such that $a\ \varepsilon\ a$ and $b\ \varepsilon\ b$, we can introduce $a \cup b$.

When we use DISTR', our EC from the beginning of this article becomes:

$$\text{J4} \qquad \forall_{a,b}(a\ \varepsilon\ el(b) \equiv \exists_c(b\ \varepsilon\ c \wedge a\ \varepsilon\ c))$$

Now we can easily prove the following:

$$\text{PAR1} \qquad \forall_{a,b}(a\ \varepsilon\ a \wedge b\ \varepsilon\ b \rightarrow a\ \varepsilon\ el(b))$$

which says that for any two existing objects one is an element of the other. The proof goes as follows:

Assume $a\ \varepsilon\ a \wedge b\ \varepsilon\ b$. It means that there are: a unique object named by a and a unique object named by b. We can introduce a name $a \cup b$. It follows from D$\cup$ that $a\ \varepsilon\ a \cup b$ and $b\ \varepsilon\ a \cup b$. Hence, $\exists_c(b\ \varepsilon\ c \wedge a\ \varepsilon\ c)$ (obviously, our $a \cup b$ is such a c). By J4, $a\ \varepsilon\ el(b)$. By repeating this reasoning for b instead of a and a instead of b, we obtain the consequence that for any two existing objects, they are elements of each other. By repeating this reasoning in a little bit simpler form, we obtain the result that

$$\text{PAR2} \qquad \forall_a(a\ \varepsilon\ a \rightarrow a\ \varepsilon\ el(a))$$

which would mean that each existing object is a *distributive* element of itself.

Still, that is not all — we can even obtain the result that the relation of 'being a distributive element of' is transitive:

$$\text{PAR3} \qquad \forall_{a,b}(a\ \varepsilon\ el(b) \wedge b\ \varepsilon\ el(c) \rightarrow a\ \varepsilon\ el(c))$$

To prove it, assume that $a\ \varepsilon\ el(b) \wedge b\ \varepsilon\ el(c)$. It implies that $a\ \varepsilon\ a$. If the name $el(c)$ is not empty, it means that there is a unique class c.[76]

[76] According to the fourth axiom of (Leśniewski, [29, pp. 270-272]):

It follows that $c\ \varepsilon\ c$. Clearly, $a\ \varepsilon\ a \vee a\ \varepsilon\ c$. It follows that $a\ \varepsilon\ a \cup c$ and $c\ \varepsilon\ a \cup c$. Hence, $\exists_d [a\ \varepsilon\ d \wedge c\ \varepsilon\ d]$ ($a \cup c$ is such a d). By J4, we obtain the consequence that $a\ \varepsilon\ el(c)$.

PAR1, PAR2, PAR3 show that while using the Leśniewski/Sobociński reduction of 'redundant' notion of distributive class, what we get is not a reduction of the notion of *distributive* class at all.

We can see that something is wrong with DISTR'. Perhaps, we should use a more intuitive claim, DISTR2?

$$\text{DISTR2} \qquad \forall_{a,b}(a\ \varepsilon\ el(Kl(b) \equiv a\ \varepsilon\ b)$$

This idea seems to be quite plausible. Nevertheless, this condition does not define the notion of class. It does not allow to get rid of the expression 'Kl' in any possible context. Moreover, DISTR2 leads to contradiction if we assume that for any a the expression '$Kl(a)$' has a designate.[77] In other words, DISTR2 not only does not do the job we would like it to do, but also its introduction requires additional restrictions put on admissible names.[78]

Let us consider the natural reading of EC:

b is an element of a iff for some c: a is the class of c and b is c.

In the collective understanding of 'class' this amounts to saying that each ingredient of a is its element. In this interpretation we obtain odd results that, say, my leg is an element of the class of men, or that I am an element of the

---

Jeżeli pewien przedmiot jest a, to pewien przedmiot jest klasą p-tów a.

which (in our language) means:

$$\text{AX IV} \qquad \forall_a(\exists_b b\ \varepsilon\ a \rightarrow \exists_c c\ \varepsilon\ Kl(a))$$

Next, according to the third axiom (*op. cit.*):

Jeżeli P jest klasą p-tów a, oraz Q jest klasą p-tów a, to P jest Q.

Which (still, in our language) means:

$$\text{AX III} \qquad \forall_{a,b,c}(c\ \varepsilon\ Kl(a) \wedge d\ \varepsilon\ Kl(a) \rightarrow c\ \varepsilon\ d)$$

[77] For a detailed analysis of Leśniewski's notion of distributive class, see (Pietruszczak, [36, pp 21-32]).

[78] Pietruszczak tries to solve this difficulty by replacing DISTR2 by two another assumptions. It is an interesting solution. It, nevertheless, has a weak point: it uses the notion of class as primitive, and the notion of 'the class of a' is defined by means of the former.

class of parts of human body. The collective understanding, though admits the implication:

$$\text{COLL1} \qquad \forall_{a,b}(a \mathrel{\varepsilon} b \rightarrow a \mathrel{\varepsilon} el(Kl(b)))$$

blocks the implication in the other direction:

$$\text{COLL2} \qquad \forall_{a,b}(a \mathrel{\varepsilon} el(Kl(b)) \rightarrow a \mathrel{\varepsilon} b)$$

On the other hand, there seems to be an equally legitimate distributive intuition that for normal names, like 'cow', 'bat', 'beer', we not only can say that a given bat (cow/beer) is an element of the class of bats (cows/beers), but also that *anything* which is an element of the class of bats (cows/beers) actually *is* a bat (cow/beer).

Following this intuition, EC seems quite plausible. What guarantees us that in the distributive interpretation COLL1 and COLL2 work, is the fact that two names which do not name exactly the same objects 'generate' different distributive classes.

5.3.2 Problems with intuition

No surprise, Leśniewski did not believe in the existence of distributive sets. Nevertheless, this view did not arise from his consideration of Russell's antinomy (perhaps psychologically it did, but logically, the antinomy does not force us to accept Leśniewski's account of sets). For him, it was rather an intuitive claim. He wrote:

> I do not know, what Mr Whitehead and Mr Russell understand in his commentaries to their system by 'class'. The circumstance that 'class' is to be, according to their position, the same object as 'extension', does not help me at all, since I do not know also, what the authors mean by "extension'... I smell in Mr Whitehead's and Russell's "classes" or Frege's "extensions of notions" the scent of mythical specimens from a rich exhibition of fictitious objects ... The difficulty of understanding what, according to Mr Russell, the difference between a 'heap' of objects a and a 'class' of objects a should consist in, is a difficulty I cannot overcome. (Leśniewski, [28, p.204-205])

The main objection Leśniewski had against the concept of distributive class was that he could not understand the concept. This, however, is not a very strong argument. It amounts to the purely psychological fact that Mr Leśniewski does not understand an expression. This sole fact does not constitute any sound argument, just as the fact that I do not understand fully what, say, strings (in physics) are, constitutes no serious argument that there are no strings, whatever being a string may consist in.

Perhaps the point of this phenomenon has been very nicely put by Leśniewski's teacher, Twardowski:

> Generally, those who follow the scheme given by Leśniewski, very arbitrarily require an analysis where they consider it comfortable; but when someone requires an analysis where they consider it uncomfortable, they refer to intuition. And when their adversary also tries sometimes to refer to intuition, they answer: 'Well, we do not understand what, according to you, should be intuitively given.' (Twardowski, [46, 12th of August 1930])

6 Syllogistic as Free Logic

6.1 Preface

Gyula Klima in his paper on reference and existence in mediaeval philosophy [20] presents a quite intuitive interpretation of categorical propositions, which allows the rules of the Square of Opposition to hold without the assumption of the non-emptiness of common names. The main purpose of this paper is three-fold: to elaborate an interpretation of categorical propositions that allegedly makes Syllogistic work as free logic, to decide which commonly accepted syllogistic rules hold both in non-empty and empty domains of objects named (both with the assumption of non-emptiness of common names, and without it), and to compare the rules thus obtained with some historical approaches to the Syllogistic of categorical propositions. The secondary purpose is to show that there are some passages in Aristotle's *Organon* as well as in some works of other logicians that the currently commonly accepted interpretation of Syllogistic fails to explain, but which can be understood on the ground of the considered interpretation. The third purpose is to define some groups of syllogistic languages and to give definitions of model, satisfaction, truth and validity for this languages.

It is important that the assumption of non-emptiness of common names (Let us denote this assumption by "AN") is not equivalent to the assumption of non emptiness of domain of objects named (Let us denote this assumption by "AD").

Let the domain of objects named be called "U". For every common name, there exists the set of its *designata*. Then, when no existence-assumption is made, the family of sets 2^U contains as its elements extensions of common names. Thus:

$AN \equiv$ the family of permitted extensions of common names is $2^U \setminus \Phi$
(only names with non-empty extensions are allowed)

whereas

$AD \equiv U \neq \Phi$ (only a non-empty domain is allowed).

Obviously, as far as we assume that some names have extensions, AN implies AD. The implication in the other direction does not hold.

6.2 Historical Background

I think it useful to distinguish at least two periods in the history of predicate calculus approach to Syllogistic.

1. During the first period logicians applied the functional calculus to Syllogistic without any additional presumptions, according to the interpretation:

$$SaP \equiv \neg\exists_x(S(x) \wedge \neg P(x)) \equiv \forall_x(S(x) \rightarrow P(x)) \tag{1}$$

$$SiP \equiv \exists_x(S(x) \wedge P(x)) \tag{2}$$

$$SeP \equiv \neg\exists_x(S(x) \wedge P(x)) \equiv \forall_x(S(x) \rightarrow \neg P(x)) \tag{3}$$

$$SoP \equiv \exists_x(S(x) \wedge \neg P(x)) \tag{4}$$

From now on, this interpretation will be called *weak* interpretation, as contrasted with the *strong* interpretation:

$$SaP \equiv \exists_x S(x) \wedge \forall_x(S(x) \rightarrow P(x)) \tag{5}$$

$$SiP \equiv \exists_x(S(x) \wedge P(x)) \tag{6}$$

$$SeP \equiv \exists_x S(x) \wedge \forall_x(S(x) \rightarrow \neg P(x)) \tag{7}$$

$$SoP \equiv \exists_x(S(x) \wedge \neg P(x)) \tag{8}$$

After such a translation, numerous rules of inference in Syllogistic appeared to be invalid. Thus, Syllogistic was believed to be an odd and old system unworthy of modern attention.

2. Later on, the value of Syllogistic began to be appreciated on the ground of its validity when some presuppositions was made; e.g. Keynes proved that Syllogistic works if we assume that there are no empty classes, and if we forbid the usage of empty common names. Similarly, Ajdukiewicz proved that it works on the only presumption that there exist at least three different objects. In these ways, the system has been rescued on the ground of *weak* or *strong* interpretation, with the help of existential presuppositions.

However, the case is not so simple.

Aristotle in [Cat. 13b 27-33] writes:[79]

> However, in the case of affirmation and negation, always, if [the object] is , and if [it] is not, one is false and the other true; For, of those [two:] that Socrates is ill and that Socrates is not ill, when Socrates exists, it is obvious that [exactly] one is true or [exclusive] false.[80]And similarly, when he does not exist: for that he is ill when he does not exist is false, and that he is not ill, is true;

Let us analyze this statement with the use of predicate calculus and the *weak* interpretation.

We represent Socrates by constant: s. Being ill by a predicate: C. Thus, the sentence: 'Socrates is ill' is: $C(s)$. Next, we define a predicate P:

$$P(x) \equiv x = s \tag{9}$$

It is clear that:

$$C(s) \equiv \exists_x [P(x) \wedge C(x)] \tag{10}$$

accordingly:

$$\neg C(s) \equiv \neg\exists_x [P(x) \wedge C(x)] \tag{11}$$

Now we claim:

$$\neg\exists_x [P(x) \wedge C(x)] \equiv \forall_x [P(x) \rightarrow \neg C(x)] \equiv PeC \tag{12}$$

(The last equivalence being obtained in *weak* interpretation),

$$PeC \rightarrow PoC \tag{13}$$

(According to the square of oppositions), and:

$$PoC \equiv \exists_x [P(x) \wedge \neg C(x)] \tag{14}$$

[79] Translations from Greek and Latin are of my own.

[80] Or we may say more clearly, less literally: when Socrates exists, exactly one is true and the other false.

(still, *weak* interpretation).

From 11, 12, 13 and 14 we obtain:

$$\neg C(s) \rightarrow \exists_x [P(x) \wedge \neg C(x)] \tag{15}$$

Now, consider the situation described by Aristotle:

1. Socrates does not exist. (*mee ontos autou*)
2. 'Socrates is ill' is false. (*to men gar nosein mee ontos pseudos*)
3. 'Socrates is not ill' is true. (*to de mee nosein aleethes*)

The proposition 2 is quite coherent with the weak interpretation. But 1 and 3 are not. For, if 3 is true, then $\exists_x[P(x) \wedge \neg C(x)]$. If it is so, it is also true, that $\exists_x P(x)$, and therefore $\exists_x(x = s)$ i.e. Socrates exists. But according to 1 he does not. Thus, we have a contradiction.

It should be clear that application of strong interpretation also results in contradiction.[81]

The argumentation given above might meet with the response that the authority of the late part of *Categoriae* is dubious, and Aristotle's understanding of the word 'essence' was such that if there is an essence of a given object x, then there also must be x. To support this objection, Aristotle himself may be quoted [An. Post. 92b 4-7]:

> Moreover, how [does one] proves the essence? For it is necessity that one who knows the essence of human, or of anything else, knows also that [it] is. (For [there is] no one who knows the essence of what is not...

Yes, I agree that Aristotle claimed it impossible to know the essence without knowing about existence. But, if it is to be a ground for an objection that Syllogistic assumed non-emptiness of terms: *non sequitur*. For aristotelian essence strictly speaking is 'what something is' (*to ti esti(n)*).In the light of this fact, it

[81] For $C(s) \equiv \exists_x[P(x) \wedge C(x)] \equiv PiC$. Now, it is not true that Socrates is ill. Therefore, it is not the case that PiC. Two subcontraries cannot both be false. Hence, it is the case that PoC. Since (in strong interpretation)$PoC \equiv \exists_x[P(x) \wedge \neg C(x)]$, it is the case that $\exists_x P(x)$, i.e. Socrates exists, which, according to Aristotle, is not the case.

seems more understandable for him to claim that one cannot know what something is if it (in general) is not. However, the claim has very little importance for the problem we investigate. For Aristotle considered the usage of terms without knowing the essence of its hypothetical designates quite legitimate. He distinguished simply between definition of a term and the cognition of an essence of a being. Thus, the passage just quoted goes on as follows:

> ...(For [there is] no one who knows the essence of what is not, but [one may know] what an expression or name means, for I can say 'buck-stag', but it is impossible to know the essence of buck-stag.

This leads us to a more general consideration. Syllogistic was believed to work when variables were replaced by names. Therefore, if we discover that Aristotle or any of his followers argues against the usage of empty names, we may say consequently that he argues against empty names in Syllogistic. And conversely, if we see that he has nothing against the usage of empty names, and moreover, argues that they may be used and that they have meaning, we have strong suggestions that, as far as he does not explicate any restrictions, he accepts the usage of empty names in Syllogistic.

That is why we shall turn our attention for a while to see that Aristotle (and not only he, but we will back to this issue soon) claimed that in scientific theories terms may be defined and used independently of whether their *designata* exist.

Aristotle believed definitions to constitute a quite important part of a theory (see e.g. An. Post. 90b). What did he say about the relation between definition and existence?

> Knowing by some definition what it is, it is unknown whether it is.[82]

and also:

> And it is clear that, according to currently used modes of definitions, those who define do not show that [designatum definiendi] exists. [83]

Thus, according to Aristotle, we may introduce into a theory terms without knowing whether their designata exist. The role that Syllogistic played in

[82][An. Post. 92b 17-18]
[83][An. Post. 92b 19-20]

his methodology strongly suggests that terms defined could take their place in syllogistic reasonings.

However, considerations of this issue are not limited only to Aristotle himself. We can find throughout the Middle Ages not only syllogistic reasonings obviously using empty terms, but also theoretical investigations about the relation between content of a name and *existentia designatorum*. Let us look at the former before coming to the latter.

Let us take under consideration *Sophismata* written by Wilhelm of Heytesbury. In 26th *sophismate* he mentions the following syllogism:

> Everything that was, is. Phoenix was. Therefore phoenix is.[84]

For us, the point is not whether there is a mistake. It is the fact that *in toto sophismate* Heytesbury never says: This syllogism is erroneous, for it uses an empty term. Moreover, he almost explicitly states that we can use terms independently of whether its designata exist. For he writes:

> ... they assume ... that any such negative particular indefinite or singular [proposition], in which the word 'is' is predicated as predicate [not as a connective], is impossible and including contradiction... [Propositions] of this kind are: 'Caesar is not', 'Chimera is not', 'It is not', when we demonstrate Socrates or Plato or anything else, 'Some fenix is not', and so about any such [propositions]. But this answer is the most ridiculous and stupid... [85]

To give an another example. Paulus Venetus in his handbook *Logica Parva* writes:

> ... this [proposition] is true: 'Some rose is not a substance', when no rose exists. But this [proposition] is false 'Some non-substance is not a non-rose', for its contradictory [proposition] is true, namely 'Every non-substance is non-rose'... [86]

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[84] [Heytesbury, Soph. 146rb]: "omne quod fuit est; fenix fuit; ergo fenix est."

[85] [Heytesbury, Soph. 146va]: "... assumunt ... quod quaelibet talis negativa particularis indefinita vel singularis in qua praedicatur hoc verbum 'est' secundum adjacens est impossibilis et includens contradictionem in se, cujusmodi sunt istae 'Caesar non est', 'chimaera non est', 'hoc non est, demonstrato Socrate vel Platone vel quocumque alio', 'aliqua fenix non est', et sic de omnibus talibus. Sed haec responsio dignissima est derisu et insipida..."

[86] [Venetus, LP, I, 36, p.10 v.28-31]: "... haec est vera 'Aliqua rosa non est substantia' nulla

Let us concentrate for a while on this reasoning in order to evaluate it on the ground of weak or strong interpretation. What Paulus says, looks in predicate calculus as follows (weak interpretation):

'Every non-substance is non-rose' is true.

$$\forall_x[\neg S(x) \to \neg R(x)] \tag{16}$$

Therefore, the contradictory proposition: 'Some non-substance is not a non-rose' is false:

$$\neg\exists_x(\neg S(x) \wedge \neg\neg R(x)] \tag{17}$$

Hitherto, everything seems to be acceptable. There are substances, there are non-substances. The first proposition is true, since every existing rose is a substance. But Paulus adds:

If there is no rose, 'Some rose is not a substance' is true:

$$\neg\exists_x R(x) \to \exists_x[R(x) \wedge \neg S(x)] \tag{18}$$

From (18) we easily obtain:

$$\neg\exists_x R(x) \to \exists_x R(x) \tag{19}$$

Of course, (19) is not yet a contradiction. However, it seems obvious that it is not the case that when no rose exists, a rose exists.

The same difficulty arises, when we apply strong interpretation, for the interpretation of 'Some rose is not a substance' is exactly the same.

Our Venetian logician writes also:

> ... there is no consequence: 'Chimera which runs, does not move. Therefore chimera runs.', because the antecedent is true, and the consequent is false... [87]

rosa existente. Et tamen haec est falsa 'Aliqua non substantia non est non rosa', quia suum contradictorium est verum, videlicet 'Omnis non substantia est non rosa'haec est vera 'Aliqua rosa non est substantia' nulla rosa existente. Et tamen haec est falsa 'Aliqua non substantia non est non rosa', quia suum contradictorium est verum, videlicet 'Omnis non substantia est non rosa' ... "

[87] [Venetus, LP, III, 8, p. 55, v. 2-4]: "... non sequitur 'Chimaera quae currit non movetur; ergo chimaera currit' quia antecedens est verum, et consequens falsum ... "

Let us treat 'Chimera which runs' as a name-expression. 'Chimera which runs, does not move.' is a negative sentence. 'Chimera runs' is a positive one. How could Paulus Venetus have known which is false and which is not? Supposedly, though he was a scientist, he hadn't observed chimeras, and noted, whether running chimeras move or not. I assume that he held the belief that there are no chimeras, without investigating whether they run or not. But if he did so (i.e. held the belief), we have a situation similar to that of ill non-existing Socrates in [Cat.]. Namely, a negative sentence (which implies some negative particular sentence with an empty subject) is true. This situation cannot occur, when either weak or strong interpretation is sound. A positive sentence, though, with an empty subject, is false, just as if positive sentences had existential import, and the negative ones had not.

Coming back to the more theoretical grounds, which we referred to by speaking about the VIIth chapter of [An. Post.]; there are two commentaries on this chapter written by two quite prominent representatives of Medieval philosophy who are sometimes even opposed to each other, namely Thomas Aquinas and Duns Scotus. Each of them has written a commentary on [An. Post.], and both seem to agree with Aristotle. Thomas writes:

> Therefore, it is not possible for anyone to prove by the same proof what [something] is, and that [it] is.[88]

Indeed, he uses this claim in practice, when he argues against the now so called 'ontological proof': [89]

> Even if anyone understood that by this name 'God' is signified what has been told, namely something such that nothing greater than it can be thought of, it would not imply that [anyone] understands that this, what is by this name signified, really exists. . . [90]

Quite interestingly, Duns Scotus devotes a particular question to the difficulty: *Utrum quaestio quid est praesupponat si est?* His answer is somewhat

[88][Aquinas, *Expositio Posteriorum*, lib. 2 l. 6 n. 3]: "Non est ergo possibile quod eadem demonstratione demonstret aliquis quid est et quia est."

[89]I find it quite awkward that the proof 'going' from words to things is called 'ontological', and not 'logoontical'.

[90][Aquinas, ST, q.II, art. 1, ad 2]: "Dato etiam quod quilibet intelligat hoc nomine Deus significari hoc quod dicitur, scilicet illud quo maius cogitari non potest; non tamen propter hoc sequitur quod intelligat id quod significatur per nomen, esse in rerum natura . . . "

more complicated:

> What [something] is presupposes neither existence of essence… nor existential existence [?]: for existential existence is firstly proper to a singular itself… but what [something] is presupposes a third existence, which is of actually being… and this existence which is nothing else than some grade of existence of one essence distinguished from the other grade of another essence… and this existence is in some way non-existence prohibited in reality; or existence in habitu… and not the existential existence… [91]

Scotus uses quite complicated language. After reading his commentary, I am non sure what *esse actualiter entis* may be. Perhaps, a thing has *esse actualiter entis* iff. it is a possible thing (it is not prohibited in nature). Nevertheless, it seems that it is not the actual existence of particular designata. First, because the existence of singulars is called esse existere; second, because it does not seem probable that the existence of singular designatorum is something which may be described by saying: this existence which is nothing else that some grade of existence of one essence distinguished from the other grade of another essence; third, because Scotus writes that the question: what [something] is…

> …presupposes some third existence which neither is the existence of essence, nor the existential existence which is measured by time… [92]

I am also quite unsure, how to understand the expression 'existence measured by time'. The first interpretation that comes to my mind is: 'existence in time'.

[91] [Duns, In Ar. Log., q. LII., p. 460]: "…quod quid est non praesupponit esse essentiae… nec praesupponit esse existere: quia esse existere est primo ipsius singularis… sed quid est praesupponit tertium esse, quod est actualiter entis… et illud esse quod est actualiter entis nihil aliud est nisi quidam gradus essendi unius essentiae distinctus contra alium gradum alterius essentiae… et illud esse est quodammodo non esse prohibitum in rerum natura; sive esse in habitu… et non esse existere… quod quid est non praesupponit esse essentiae… nec praesupponit esse existere: quia esse existere est primo ipsius singularis… sed quid est praesupponit tertium esse, quod est actualiter entis… et illud esse quod est actualiter entis nihil aliud est nisi quidam gradus essendi unius essentiae distinctus contra alium gradum alterius essentiae… et illud esse est quodammodo non esse prohibitum in rerum natura; sive esse in habitu… et non esse existere…"

[92] [Duns, In Ar. Log., q. LII., p. 461]: "…praesupponit enim aliquod tertium esse quod nec est esse essentiae nec esse existere quod mensuratur tempore …"

If it is correct, then the most probable interpretation of his (Scotus') view is that *esse existere* which is the existence of particular being is not presupposed by what [something] is. The only existence presupposed is some *esse actualiter entis* of which we know that it is: atemporal, some kind of possibility of real existence (*esse existentiae in habitu*?), and some grade of existence of one essence distinguished from the other grade of another essence.

To recapitulate the historical part of this paper: There are some passages in Aristotle and Medieval logicians that are incoherent with the two currently accepted interpretations of assertoric Syllogistic. On the grounds of their (logicians') own words it would be somewhat illegitimate to consider that they excluded from Syllogistic the usage of empty terms.

6.3 Systematic Approach

6.3.1 Syllogistic

By "Syllogistic" in this paper I denote only that part of Medieval logic which is concerned with simple, categorical propositions and relations between them (the term is well-known enough to make the definition redundant; the emphasis is only put to exclude analysis of modalities or intensional contexts).

There are, respectively to the rules of inference accepted, at least a few systems of syllogistic.

6.3.2 Wide Syllogistic (W-S)

To denote the fullest (containing the longest list of immediate inferences) syllogistic, I use the abbreviation: "W-". W-S consists of following rules of immediate inference:[93]

[93] In the presentations of Syllogistic below some simplifications have been committed; However, in any syllogistic besides W-S I tried to list those rules which both: were accepted by a given logician and belong to W-S.

SaP	SiP	SeP	SoP	RULE
SeP'	SoP'	SaP'	SiP'	Obversion
PiS	PiS	PeS	—	Conversion
PoS'	PoS'	PaS'	—	Obversion of the result of conversion
$P'eS$	—	$P'iS$	$P'iS$	Partial contraposition
$P'aS'$	—	$P'oS'$	$P'oS'$	Complete contraposition
$S'oP$	—	$S'iP$	—	Partial inversion
$S'iP'$	—	$S'oP'$	—	Complete inversion

- all rules of the Square of Opposition

- all syllogisms: Barbara, Celarent, Darii, Ferio Cesare, Camestres, Festino, Baroco Darapti, Disamis, Datisi, Felapton, Bocardo, Ferison Bamalip, Camenes, Dimatis, Fesapo, Fresison

6.3.3 Aristotle's Syllogistic (A-S)

1. Conversion[94] : of SaP (partial), of SiP (complete) and of SeP (complete)
2. All rules of the Square of Opposition[95]
3. All syllogisms

6.3.4 Peter of Spain's Syllogistic (H-S)

It consists of following rules: [96]

1. Simple conversion of SeP and SiP
2. Conversion *per accidens* of SeP and SaP
3. Complete contraposition of SaP and SoP
4. Square of Opposition
5. Syllogisms

[94] The description of this Syllogistic is especially explained in An. Pr.. Quite a precise presentation is also to be find in [4, pp. 42-54] and in [32, pp. 34-104].

[95] Square of Opposition is described rather in his work De. Int..

[96] The description of his syllogistic to be found in [16, I.18-I.21].

6.3.5 Ockham's Syllogistic (O-S)

1. Simple[97] conversion of SeP and SiP
2. Conversion *per accidens* of SaP
3. Square of opposition
4. Syllogisms

6.3.6 Paulus of Venice's Syllogistic (P-S)

1. Simple[98] conversion of SeP, SiP
2. Conversion *per accidens* of SaP
3. Contraposition of SaP, SoP
4. Obversion (at least of SaP and SeP)
5. Square of Opposition
6. Syllogisms

6.4 K- Interpretation

Gyula Klima in his paper [20] presents a quite different interpretation of truth-conditions of categorical propositions:

> According to what historians of Medieval logic dubbed the inherence theory of predication, an affirmative categorical proposition

[97] Description of his syllogistic in [34, II.]. Ockham's treatment of conversion seems to me somewhat odd. First, he defines kinds of conversion - simplex, *per accidens*, and contraposition. Then, he gives rules for the first two only, leaving contraposition aside. Thus, I list only those rules, that were by Ockham explicitly stated.

[98] The Syllogistic of Paulus of Venice does not differ very much from those of Buridan, besides only one rule, given by Paulus, and obtainable from the rules given by Buridan (those rules to be found in Buridan's *Summulae de Dialectica*). Strictly speaking, both, Buridan and Paulus, give such rules of equipollence: $SaP' \equiv SeP$, $SeP' \equiv SaP$, while in Buridan the rules given for SaP' and SeP' seem to be examples of a general rule. (In Buridan it is the second rule of equipollence, Paulus writes about it in I-st tractatus, cap. 10 *De aequipollentis*). As to Buridan, I use the version I cannot quote or refer to. As to Paulus Venetus, I use [48].

> (in the present tense with no ampliation[...]) is true only if an individualized property (form, or nature) signified by the predicate term actually inheres in the thing(s) referred to by the subject term.
>
> On the other hand, according to the other basic type of Medieval predication theories, the so-called identity theory, an affirmative categorical proposition is true only if its subject and predicate terms refer to the same thing or things. For example, on this analysis 'Socrates is wise' is true if and only if Socrates, the referent of 'Socrates', is one of the wise persons, the referents of the term 'wise'. If any of the two terms of an affirmative categorical is "empty", then the term in question refers to nothing. But then, since "nothing is identical with or diverse from a non-being", as Buridan (the "arch-identity-theorist" of the 14th century) put it...

Farther on he presents an account which, I think, allows us to construct the following interpretation of categorical propositions in predicate calculus:

$$SaP \equiv \exists_x S(x) \wedge \forall_x (S(x) \rightarrow P(x)) \tag{20}$$

$$SiP \equiv \exists_x (S(x) \wedge P(x)) \tag{21}$$

$$SeP \equiv \forall_x (S(x) \rightarrow \neg P(x)) \tag{22}$$

$$SoP \equiv \neg \exists_x S(x) \vee \exists_x (S(x) \wedge \neg P(x)) \tag{23}$$

Let us denote this interpretation K-Interpretation.

In his paper Klima has shown that in this interpretation the Square of Opposition holds for empty common names as subjects. The purpose of the following deliberations is to decide what rules of syllogistic (W-S) hold for empty common names (as subjects and as predicates), as well as in empty domain of objects named, and to compare the resulting set of rules (let us call it F-S from "Free") with W-S, A-S, H-S, O-S, and V-S introduced above.

6.5 Decision Procedure

In deciding the above question we will apply the method called (in Poland, at least) 0-1 Decision Procedure for Narrower 1-place Predicate Calculus (as described, e.g. in [7, 114-126]). For convenience, in this paper we give only the results of this procedure. The procedure itself is quite uncomplicated, mechanical, boring, and (what justifies the omission) repeatable.

In general, we start with forming implications corresponding to a given rule, and translate such a formula of Syllogistic into predicate calculus according to K-interpretation. Then, we check the validity of the formula obtained by the above-mentioned decision procedure.

We denote the set that is the extension of a given name, say S, by $D(S)$.

6.5.1 Obversion

1. The formula:
$$SaP \to SeP' \tag{24}$$
is valid.

2. The formula
$$SiP \to SoP' \tag{25}$$
is valid.

3. The formula
$$SeP \to SaP' \tag{26}$$
is invalid. It is falsified when $D(S) \cap D(P) = D(S) \cap D(P') = \Phi$.

4. The formula
$$SoP \to SiP' \tag{27}$$
is invalid. It is falsified when $D(S) \cap D(P) = D(S) \cap D(P') = \Phi$.

6.5.2 Conversion

1. The formula
$$SaP \to PiS \tag{28}$$
is valid.

2. The formula
$$SiP \to PiS \tag{29}$$
is valid.

3. The formula
$$SeP \to PeS \tag{30}$$
is valid.

6.5.3 Obversion of the result of Conversion

1. The formula
$$SaP \to PoS' \tag{31}$$
is valid.

2. The formula
$$SiP \to PoS' \tag{32}$$
is valid.

3. The formula
$$SeP \to PaS' \tag{33}$$
is invalid. It is falsified when $D(S) \cap D(P) = D(S) \cap D(P') = \Phi$.

6.5.4 Partial Contraposition

1. The formula
$$SaP \to P'eS \tag{34}$$
is valid.

2. The formula
$$SeP \to P'iS \tag{35}$$
is invalid. It is falsified when $D(S) \cap D(P) = D(S) \cap D(P') = \Phi$.

3. The formula
$$SoP \to P'iS \tag{36}$$
is invalid. $D(S) \cap D(P) = D(S) \cap D(P') = \Phi$.

6.5.5 Complete Contraposition

1. The formula
$$SaP \to P'aS' \tag{37}$$
is invalid. It is falsified when $D(S) \cap D(P) \neq \Phi$ and $D(S) \cap D(P') =$ $= D(S') \cap D(p') = \Phi$.

2. The formula
$$SeP \to P'oS' \tag{38}$$
is invalid. It is falsified when $D(S) \cap D(P) = D(S) \cap D(P') = \Phi$ and $D(S') \neq D(P')$.

3. The formula
$$SoP \to P'oS' \tag{39}$$
is invalid. It is falsified when $D(S) \cap D(P) = D(S) \cap D(P') = \Phi$ and $D(S') \cap D(P') \neq \Phi$.

6.5.6 Partial Inversion

1. The formula
$$SaP \to S'oP \tag{40}$$
is invalid. It is falsified when the following conditions are fulfilled:
   (a) $D(S) \cap D(P') = D(S') \cap D(P') = \Phi$
   (b) $D(S) \cap D(P) \neq \Phi$
   (c) $D(S') \cap D(P) \neq \Phi$

2. The formula
$$SeP \to S'iP \tag{41}$$
is invalid. It is falsified when $D(S) \cap D(P) = D(S') \cap D(P) = \Phi$.

6.5.7 Complete Inversion

1. The formula
$$SaP \to S'iP' \tag{42}$$
is invalid. It is falsified when $D(S) \cap D(P) \neq \Phi$ and $D(S) \cap D(P') =$ $D(S') \cap D(P') = \Phi$.

2. The formula
$$SeP \to S'oP' \tag{43}$$
is invalid. It is falsified when $D(S) \cap D(P) = D(S') \cap D(P) = \Phi$ and $D(S) \cap D(P') \neq \Phi$.

6.5.8 Square of Opposition

All rules of the Square are valid.

6.5.9 Syllogisms

All syllogisms are valid.

6.6 Comparison

Below, I construct a table in which a comparison of A-S, W-S, H-S, O-S, P-S, and F-S is conducted. By "+" I denote the fact that a rule holds, and that it does not hold by "-". In the case of a given historical syllogistic, no sign indicates that the author does not mention the rule explicitly, neither accepts nor refutes it. The first column of the table gives the traditional name of the rule. The second indicates the kind of a proposition that is the antecedent of a given rule (A, E, I, O). The third column indicates the number of formula under consideration (according to the numeration given in this paper).

RULE	ANT.	FOR.	W-S	A-S	H-S	O-S	P-S	F-S
Obversion	A	(24)	+				+	+
	I	(25)	+					+
	E	(26)	+				+	-
	O	(27)	+					-
Conversion	A	(28)	+	+	+	+	+	+
	I	(29)	+	+	+	+	+	+
	E	(30)	+	+	+	+	+	+
Obversion of the result of Conversion	A	(31)	+					+
	I	(32)	+					+
	E	(33)	+					-
Partial Contraposition	A	(34)	+					+
	E	(35)	+					-
	O	(36)	+					-
Complete Contraposition	A	(37)	+		+		+	-
	E	(38)	+					-
	O	(39)	+		+		+	-
Partial Inversion	A	(40)	+					-
	E	(41)	+					-
Complete Inversion	A	(42)	+					-
	E	(43)	+					-
Square	—	—	+	+	+	+	+	+
Syllogisms	—	—	+	+	+	+	+	+

6.7 Models For Syllogistic

6.7.1 Preliminaries

To this point we have developed the idea given by Gyula Klima than there is an interpretation in which syllogistic may be considered as a free logic (i.e. a logic which works without a presumption of existence of any objects). We have shown exactly which commonly accepted syllogistic rules are valid in empty domain and compared the result with some classic representatives of Syllogistic. Our present purpose is to define a model for Syllogistic and for Syllogistic which is free in K-interpretation (hence K-MODEL). We will also try to define the interpretation of a language of Syllogistic in a model, on the grounds of formal semantics. Therefore, the present task is to provide mainly a notational (and notional) apparatus.

CONVENTION 1 *We introduce the following metalanguage variables:*

variables representing propositions: $\varphi, \psi, \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$

variables representing propositional formulas: $\tau, \tau_1, \ldots, \tau_n$

variables representing propositional expressions: $\chi, \chi_1, \ldots, \chi_n$

variables representing names: $\mu, \nu, \mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n$

variables representing name variables $\pi, \pi_1, \ldots, \pi_n$

variables representing name expressions $\alpha, \beta, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$

variables representing languages $L, L_1, \ldots, L_n$

variables representing sets $X, Y, X_1, \ldots, X_n, Y_1, \ldots, Y_n$

The difference between propositional formulas and propositions (as well as between name formulas and names) consists in the occurrence of free variables in the first, and their absence in the second.

6.7.2 S_{SEN} Languages

We shall start by defining the set of languages of assertoric sentences (S_{SEN}). They are languages without variables. We shall not distinguish between languages and its algebras of expressions.

Definition 52 $L \in S_{SEN}$ *iff.* $L = \langle W^L_{SEN}, N^L_{SEN}, a, e, o, i, \neg, \wedge \rangle$, *where*

1. N^L_{SEN} *is a set of names occurring in L.*

2. *The symbols* a, e, o, i *are functors of category* $\frac{s}{n,n}$.

3. *The symbols* $\neg, \wedge$ *are the classical functors of negation and conjunction.*

4. W^L_{SEN} *is the set of sentences, dependent on the set* N^L_{SEN}. *It is the least set which fulfills the conditions (44) and (45):*

$$\mu, \nu \in N^L_{SEN} \rightarrow \ulcorner \mu a \nu \urcorner, \ulcorner \mu e \nu \urcorner, \ulcorner \mu i \nu \urcorner, \ulcorner \mu o \nu \urcorner \in W^L_{SEN} \tag{44}$$

$$\varphi, \psi \in W^L_{SEN} \rightarrow \ulcorner \neg\varphi \urcorner, \ulcorner \varphi \wedge \psi \urcorner \in W^L_{SEN} \tag{45}$$

In other words, a language of assertoric sentences is the set of all categorical propositions built from names of this language and functors a, e, i, o, closed under the operation of conjunction and negation.[99]

Definition 53 *The set of categorical sentences of a given language $L \in S_{SEN}$ (denoted by "CAT^L_{SEN}") is the set dependent on the set N^L_{SEN}. It is the least set which fulfills the condition (44) only (obviously, $CAT^L_{SEN} \subseteq W^L_{SEN}$).*

Definition 54 *The set of hypothetical sentences of a given language $L \in S_{SEN}$ (denoted by "HYP^L_{SEN}") is the set: $W^L_{SEN} \setminus CAT^L_{SEN}$.*

Definition 55 *$\mathcal{M}$ is a model for $L \in S_{SEN}$ iff. $\mathcal{M} = \langle Ob, \{1,0\}, A, E, I, O \rangle$, where Ob is a set of objects, $\{1,0\}$ are possible values of sentences, and A, E, O, I are relations between elements of 2^{Ob}. An occurrence of such a relation between sets X, Y will be denoted eg.: $A(X,Y)$ or $I(X,Y)$.*

Definition 56 *If the model fulfills additionally the conditions (46), (47), (48), (49):*

$$\forall_{X,Y \in 2^{Ob}}[A(X,Y) \equiv X \neq \Phi \wedge X \subseteq Y] \tag{46}$$

$$\forall_{X,Y \in 2^{Ob}}[E(X,Y) \equiv X \cap Y = \Phi] \tag{47}$$

$$\forall_{X,Y \in 2^{Ob}}[I(X,Y) \equiv X \cap Y \neq \Phi] \tag{48}$$

$$\forall_{X,Y \in 2^{Ob}}[O(X,Y) \equiv X = \Phi \vee X \setminus Y \neq \Phi] \tag{49}$$

then it is called K-model *(since then it is a model of a Syllogistic free in K-interpretation).*

Let us assume that the language L and the model $\mathcal{M}$ are fixed. Now we define the set of functions of denotation[100] DEN^L_{SEN} mapping N^L_{SEN} into 2^{Ob}.

[99]Obviously, all other functors of propositional calculus can be introduced by a definition. For the sake of simplicity, we introduced as few functors, as possible. In this paper, the names "proposition" and "sentence" are exchangeable, their diversity being introduced also for convenience.

[100]I do interpret the name 'denotation' exactly according to the definition, though it may differ from the common use of this term.

Definition 57 *D^L_{SEN} is a function of denotation in a language L in a model $\mathcal{M}$ iff. it is a function on names occurring in this language, into the power set of the set of objects.*

$$D^L_{SEN} \in DEN^L_{SEN} \equiv \forall_{\mu \in N^L_{SEN}} \exists!_{X \in 2^{Ob}} [D^L_{SEN}(\mu) = X] \tag{50}$$

We define now the set of valuations of sentences in L assuming that D^L_{SEN} is fixed (since we have defined L quite syntactically, it does not have to be so). [101]

Definition 58 *$V^L_{SEN} \in VAL^L_{SEN}$ iff. the following conditions are fulfilled:*

1. *V^L_{SEN} is a function mapping W^L_{SEN} into $\{1, 0\}$.*
2. *If $\varphi \in CAT^L_{SEN}$ then:*
   (a) $V^L_{SEN}(\ulcorner \mu a \nu \urcorner) = 1 \equiv A(D^L_{SEN}(\mu), D^L_{SEN}(\nu))$
   (b) $V^L_{SEN}(\ulcorner \mu e \nu \urcorner) = 1 \equiv E(D^L_{SEN}(\mu), D^L_{SEN}(\nu))$
   (c) $V^L_{SEN}(\ulcorner \mu i \nu \urcorner) = 1 \equiv I(D^L_{SEN}(\mu), D^L_{SEN}(\nu))$
   (d) $V^L_{SEN}(\ulcorner \mu o \nu \urcorner) = 1 \equiv O(D^L_{SEN}(\mu), D^L_{SEN}(\nu))$
3. *If $\varphi, \psi \in W^L_{SEN}$ then:*
   (a) $V^L_{SEN}(\ulcorner \neg\varphi \urcorner) = 1 \equiv V^L_{SEN}(\varphi) = 0$
   (b) $V^L_{SEN}(\ulcorner \varphi \wedge \psi \urcorner) = 1 \equiv V^L_{SEN}(\varphi) = 1 \wedge V^L_{SEN}(\psi) = 1$

It could be claimed that interpretation of a language consists in model, denotation and valuation. However, in the case of valuation of sentences of S_{SEN} languages, as far as relations A, E, IO of $\mathcal{M}$ are fixed, there can be exactly one valuation of a given formula (what corresponds with the intuition that a proposition has exactly one value). Therefore, we can say:

Definition 59 *An interpretation of a given S_{SEN} language L is a pair consisting of a model and a denotation function.*

$$I^L_{SEN} \in INT^L_{SEN} \equiv \exists_{\mathcal{M}, D^L_{SEN}} [I^L_{SEN} = \langle \mathcal{M}, D^L_{SEN} \rangle] \tag{51}$$

[101] Anyway, the definition can be relativized to a given D^L_{SEN}, *mutatis mutandis.*

Now, the definition of truth (in interpretation I^L_{SEN}) for propositions of S_{SEN} languages is quite trivial.

Definition 60 $T_{I^L_{SEN}}(\varphi) \equiv \forall_{V^L_{SEN}}[V^L_{SEN}(\varphi) = 1]$

6.7.3 S_{EXP} Languages

Now, we are going to define the set of those languages of Syllogistic which contain constant names and name variables: S_{EXP} . This languages seems to be the closest to the Syllogistic as it was used in the Middle Ages, since it was a theory introduced in common language containing names and name variables.

Definition 61 $L \in S_{EXP}$ *iff.* $L = \langle E^L_{EXP}, NV^L_{EXP}, a, e, o, i, \neg, \wedge\rangle$, *where*

1. $N^L_{EXP} \neq \Phi$ *is a set of names occurring in* L.
2. $N^L_{EXP} \subset NV^L_{EXP}$
3. *The symbols* a, e, o, i *are functors of category* $\frac{s}{n,n}$.
4. *The symbols* $\neg, \wedge$ *are the classical functors of negation and conjunction.*
5. $P^L_{EXP} \neq \Phi$ *is the set of sentences, dependent on the set* N^L_{EXP}. *It is the least set which fulfills the conditions (52) and (53):*

$$\mu, \nu \in N^L_{EXP} \to \ulcorner \mu a \nu \urcorner, \ulcorner \mu e \nu \urcorner, \ulcorner \mu i \nu \urcorner, \ulcorner \mu o \nu \urcorner \in P^L_{EXP} \tag{52}$$

$$\varphi, \psi \in P^L_{EXP} \to \ulcorner \neg\varphi \urcorner, \ulcorner \varphi \wedge \psi \urcorner \in P^L_{EXP} \tag{53}$$

6. $P^L_{EXP} \subset E^L_{EXP}$
7. $NV^L_{EXP} \setminus N^L_{EXP} = \{\pi_1, \ldots, \pi_n\} \neq \Phi$, *where* $\{\pi_1, \ldots, \pi_n\}$ *is the set of all name variables of a given language* L, *i.e.* VAR^L_{EXP}.
8. E^L_{EXP} *is the least set satisfying the conditions (54), (55):*

$$\alpha, \beta \in NV^L_{EXP} \to \ulcorner \alpha a \beta \urcorner, \ulcorner \alpha e \beta \urcorner, \ulcorner \alpha i \beta \urcorner, \ulcorner \alpha o \beta \urcorner \in E^L_{EXP} \tag{54}$$

$$\chi_1, \chi_2 \in E^L_{EXP} \rightarrow \ulcorner \neg\chi_1 \urcorner, \ulcorner \chi_1 \wedge \chi_2 \urcorner \in E^L_{EXP} \tag{55}$$

In other words, a language of assertoric expressions (formulas as well as sentences) is the set of all categorical expressions built from names and name variables of this language and functors a, e, i, o, closed under the operation of conjunction and negation.

We can define the set of formulas of a given language:

Definition 62

$$FOR^L_{EXP} = E^L_{EXP} \setminus P^L_{EXP} \tag{56}$$

Definition 63 *The set of categorical expressions of a given language $L \in S_{EXP}$ (denoted by "CAT^L_{EXP}") is the set dependent on the set NV^L_{EXP}. It is the least set which fulfills the condition (54) only (obviously, $CAT^L_{EXP} \subset E^L_{EXP}$).*

Definition 64 *The set of hypothetical expressions of a given language $L \in S_{EXP}$ (denoted by "HYP^L_{EXP}") is the set: $E^L_{EXP} \setminus CAT^L_{EXP}$.*

We can introduce a notion of an expressional extension of a language belonging to S_{SEN}. Namely, we can say that:

Definition 65 *Language $L_2 \in S_{EXP}$ is an expressional extension of a language $L_1 \in S_{SEN}$ iff.*

$$P^{L_2}_{EXP} = W^{L_1}_{SEN} \wedge N_{EXP}L_2 = N^{L_1}_{SEN} \tag{57}$$

The notion of a model remains generally the same, as in the case of S_{SEN} languages, so, *mutatis mutandis*, we can apply the DEFINITION 55 on p. 111 (with all the remarks added there).

As the definition of the set of denotation functions DEN^L_{EXP} mapping NV^L_{EXP} into 2^{Ob} we can, *mutatis mutandis*, accept DEFINITION 57 on p. 112.

We define now the set of valuations of propositional expressions in L for a given D^L_{EXP}.

Definition 66 $V^L_{EXP} \in VAL^L_{EXP}$ *iff. the following conditions are fulfilled:*

1. *V^L_{EXP} is a function mapping E^L_{EXP} into $\{1, 0\}$.*

2. *If $\chi \in CAT^L_{EXP}$ then:*

(a) $V^L_{EXP}(\lceil \alpha a \beta \rceil) = 1 \equiv A(D^L_{EXP}(\alpha), D^L_{EXP}(\beta))$

(b) $V^L_{EXP}(\lceil \alpha e \beta \rceil) = 1 \equiv E(D^L_{EXP}(\alpha), D^L_{EXP}(\beta))$

(c) $V^L_{EXP}(\lceil \alpha i \beta \rceil) = 1 \equiv I(D^L_{EXP}(\alpha), D^L_{EXP}(\beta))$

(d) $V^L_{EXP}(\lceil \alpha o \beta \rceil) = 1 \equiv O(D^L_{EXP}(\alpha), D^L_{EXP}(\beta))$

3. *If* $\chi_1, \chi_2 \in E^L_{EXP}$ *then:*

(a) $V^L_{EXP}(\lceil \neg \chi_1 \rceil) = 1 \equiv V^L_{EXP}(\chi_1) = 0$

(b) $V^L_{EXP}(\lceil \chi_1 \wedge \chi_2 \rceil) = 1 \equiv V^L_{EXP}(\chi_1) = 1 \wedge V^L_{EXP}(\chi_2) = 1$

For a given denotation there can be exactly one valuation of a given formula. Therefore we define interpretation, as in DEFINITION 59 on p. 112 (obviously, when necessary changes are made).

The definition of truth (in interpretation I^L_{SEN}) for propositional expressions of S_{SEN} languages is little more complicated. For sentences, we may simply accept DEFINITION 60 from p. 113.

Some doubts arise, however, when we want to consider name variables. For we can either emphasize that they are NAME variables, or that they are name VARIABLES. The question is: should we valuate a name variable *via* names, or not? If we choose the first option, consequently, we can allow only those values od denotation for variables as arguments, which can be 'obtained' by means of substitution of name variables by names as well.

Namely, we must agree that, when we understand a valuation of k free name variables of given expression $\langle \pi_1, \ldots, \pi_k \rangle$ as a sequence $\langle X_1, \ldots, X_k \rangle$ of elements of 2^{Ob} (X_i being $D(\pi_i)$), we have to exclude from possible valuations such X_i, for which $\neg \exists_{\mu \in N^L_{EXP}} [X_i = D(\mu)]$.

We could avoid this difficulty by the simple assumption that any $L \in S_{EXP}$ is such, that

$$\forall_{X \in 2^{Ob}} \exists_{\mu \in N^L_{EXP}} [X = D(\mu)] \tag{58}$$

Unfortunately, languages which do not fulfill the condition 58) seem to be quite legitimate objects of investigation.

For convenience, we have decided to define valuation for S_{EXP} languages by means of substitution, and to leave the most general concept of valuation for S_{FOR} languages which do not contain names. There is no loss of accuracy, since all formulas valid in S_{EXP} languages according to the former notion of valuation of name variables are also valid according to the latter. We will come back to the comparison of these notions of tautology after introducing them both.

"We shall define the complete-name-substitution. Only free[102] name variables can be substituted (hence, "name-substitution"), and all variables in a formula are substituted (hence, "complete").

Definition 67 *An expression χ in which all name variables are $\pi_1, \ldots, \pi_k$ (and names are: $\nu_1, \ldots, \nu_c$) yields as a result of* complete name substitution $(\pi_1/\mu_1, \cdots, \pi_k/\mu_k)$ *a sentence φ, which contains names: $\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_c$ iff. the sentence φ differs from the expression χ only in having the names $\mu_1, \ldots, \mu_k$ in all those places in which, respectively, the name variables $\pi_1, \ldots, \pi_k$ occur in χ.*

We define the set of valuations of all name variables in $L \in S_{EXP}$: $VVAL^L_{EXP}$:

Definition 68 *If $\langle \pi_1, \ldots, \pi_k \rangle$ is the sequence of all elements of VAR^L_{EXP}, then any k-place sequence s^L_k of names belonging to N^L_{EXP} is an element of $VVAL^L_{EXP}$.*

Now we define s^L_k-substitution, which is a kind of complete name substitution. If the sequence of all elements of VAR^L_{EXP} , $\langle \pi_1, \ldots, \pi_n \rangle$, and valuation s^L_k are fixed:

Definition 69 *An expression χ, in which all name variables are $\pi_1, \ldots, \pi_k$ $(k \leq n)$ yields as the result of s^L_k - substitution a sentence varphi which contains at least names $\mu_1, \ldots, \mu_k$ iff. the sentence φ differs from the formula χ only in having the names $\mu_1, \ldots, \mu_k$ in all those places in which the name variables $\pi_1, \ldots, \pi_k$ respectively occur in χ, and, for every i, if π_i is m-th in the sequence $\pi_1, \ldots, \pi_k$, then μ_i is m-th in the sequence s^L_k .*

If s^L_k is given, we denote the result of s^L_k substitution in χ by "$s^L_k(\chi)$. It remains to define the set of valid expressions of $L \in S_{EXP}$ i.e. VAL^L_{EXP} :

[102]In the languages under investigation all name variables are free in the place of their occurrence.

Definition 70

$$\chi \in VAL^{L}_{EXP} \equiv \chi \in FOR^{L}_{EXP} \wedge \forall_{s^{L}_{k}, I}[T_{I^{L}_{EXP}}(sk^{L}_{k}(\chi))] \tag{59}$$

The objects of our investigation are languages of assertoric Syllogistic containing some name expressions. If a language fulfill this condition, then it either contains only names as name expressions, or both names and name variables, or only variables. The first two languages have been considered above. It remains to define languages both with name variables and without names.

6.7.4 S_{FOR} Languages

Definition 71 *$L \in S_{FOR}$ iff. $L = \langle W^{L}_{FOR}, N^{L}_{FOR}, a, e, o, i, \neg, \wedge\rangle$, where restrictions regarding $a, e, o, i, \neg, \wedge$ are the same as in the case od DEFINITION 52 on p. 110 , and:*

1. *N^{L}_{FOR} is a set of name variables occurring in L.*
2. *W^{L}_{FOR} is the set of formulas, dependent on the set N^{L}_{FOR}. It is the least set which fulfills the conditions (60) and (61):*

$$\pi_1, \pi_2 \in N^{L}_{FOR} \rightarrow \ulcorner \pi_1 a \pi_2 \urcorner, \ulcorner \pi_1 e \pi_2 \urcorner, \ulcorner \pi_1 i \pi_2 \urcorner, \ulcorner \pi_1 o \pi_2 \urcorner \in W^{L}_{FOR} \tag{60}$$

$$\tau_1, \tau_2 \in W^{L}_{FOR} \rightarrow \ulcorner \neg \tau_1 \urcorner, \ulcorner \tau_1 \wedge \tau_2 \urcorner \in W^{L}_{FOR} \tag{61}$$

Definition 72 *The set of categorical formulas of $L \in S_{FOR}$ CAT^{L}_{FOR} is the least set satisfying only the condition (60).*

Definition 73

$$HYP^{L}_{FOR} = W^{L}_{FOR} \setminus CAT^{L}_{FOR} \tag{62}$$

The definition of model is almost the same, as in DEFINITION 55 (*mutatis mutandis*).

The definition of valuation of name variables of a given $L \in S_{FOR}$ is a little bit different from DEFINITION 68 on p. 116 .

Definition 74 *The sequence s^{L}_{k} is a valuation of name variables of a language $L \in S_{FOR}$ iff. $\exists_{X_1,\ldots,X_n}[s_k = \langle X_1, \ldots, X_n\rangle]$*

Now we define the satisfaction of a formula τ of a given language L in a model $\mathcal{M}$ by a sequence (in a valuation) $s_k = \langle X_1, \ldots, X_n \rangle$.[103]

Definition 75 *Formula τ of $L \in S_{FOR}$ is satisfied by the sequence s_i (i.e. in valuation s_i) in the model $\mathcal{M}$, formally: $STSF^L_{s_i}(\tau)_\mathcal{M}$ iff. the following conditions are fulfilled*

1. *If $\tau \in CAT^L_{FOR}$, then:*
   (a) $\tau = \ulcorner \pi_i a \pi_j \urcorner \rightarrow [STSF^L_{s_i}(\tau)_\mathcal{M} \equiv A(X_i, X_j)]$
   (b) $\tau = \ulcorner \pi_i e \pi_j \urcorner \rightarrow [STSF^L_{s_i}(\tau)_\mathcal{M} \equiv E(X_i, X_j)]$
   (c) $\tau = \ulcorner \pi_i i \pi_j \urcorner \rightarrow [STSF^L_{s_i}(\tau)_\mathcal{M} \equiv I(X_i, X_j)]$
   (d) $\tau = \ulcorner \pi_i o \pi_j \urcorner \rightarrow [STSF^L_{s_i}(\tau)_\mathcal{M} \equiv O(X_i, X_j)]$
2. *If $\tau, \tau_1, \tau_2 \in W^L_{EXP}$ then:*
   (a) $\tau_1 = \ulcorner \neg \tau_2 \urcorner \rightarrow [STSF^L_{s_i}(\tau_1)_\mathcal{M} \equiv \neg STSF^L_{s_i}(\tau_2)_\mathcal{M}]$
   (b) $\tau = \ulcorner \tau_1 \wedge \tau_2 \urcorner \rightarrow [STSF^L_{s_i}(\tau)_\mathcal{M} \equiv STSF^L_{s_i}(\tau_1)_\mathcal{M} \wedge STSF^L_{s_i}(\tau_2)_\mathcal{M}]$

Next, we can define the expression: "The formula τ of $L \in S_{FOR}$ holds in the model $\mathcal{M}$, formally: $HLD^L(\tau)_\mathcal{M}$:

Definition 76

$$HLD^L(\tau)_\mathcal{M} \equiv \forall_{s_i} STSF^L_{s_i}(\tau)_\mathcal{M} \tag{63}$$

This notion introduced, the definition of the set od valid formulas of $L \in S_{FOR}$ (VAL^L_{FOR}) is quite trivial:

Definition 77

$$\tau \in VAL^L_{FOR} \equiv \forall_\mathcal{M}[HLD^L(\tau)_\mathcal{M}] \tag{64}$$

The definitions we introduced are very general. Especially in common use, only particular kinds of models are taken under consideration. For example, if we accept K-Interpretation of categorical propositions, we will be only interested in K-models, and we will relativize the set of valid formulas to this set of models, consequently claiming that the formula is tautology iff. it holds in any K-model.

[103] It is clear, that is is 4-place relation.

If we choose any other interpretation of categorical proposition (formally equivalent to choosing four relations between sets: A, E, I, O), the corresponding set of models in which we will be interested will also change (in a quite obvious manner, it will be restricted to those models, in which relations A, E, I, O in 2^{Ob} are exactly the set-theoretic relations we have chosen).

7 Summary

We have presented the well-known Leśniewski's symbols for sentence-forming functors of 1- and 2- place propositional arguments, and developed a general method of representing any $n \geq 3$-place functors in accordance with this notation. By doing so, we have also shown that all n-place $\frac{s}{s_1,\ldots,s_n}$ functors can be defined in Elementary Protothetic by means of the primitive basis of this system (since in any of its formulations, all 1- and 2- place functors can be defined by means of the primitive basis).

We introduced a semantic framework which allowed us to prove a kind of functional completeness of LEO. We have proposed quite an intuitive understanding of the expression 'an axiom determines the meaning of the only specific constant occurring in it' and 'the meaning of a given constant is determined axiomatically'. We have introduced some basic semantics for functors of category $\frac{s}{n,n}$ of Leśniewski's Ontology. Using this results we have proven that the popular claim that axioms of Ontology determine the meaning of primitive constants (functors), or that an axiomatic basis of Ontology determines the meaning of the primitive constant of this basis, is false.[104]

Next, we discussed some difficulties in a Leśniewskian approach to Russell's paradox. We have briefly presented Leśniewski's solution to Russell's antinomy, as formalized by Sobociński and briefly elaborated on one aspect of this solution, i.e. on the problems connected with the distributive notion of class. We have shown that the notion of distributive class which helps to solve the paradox

[104] As far as we do not require completeness with respect to a given non-standard interpretation to be a necessary condition of 'being a theory of' the functor in this interpretation. In fact, our approach which omits this condition seems to be quite plausible — at least as far, as we claim that standard arithmetics, in some sense, is a theory of numbers and some operations on them: addition, etc. (though we seem to have some problems with the completeness of arithmetics).

leads to odd conclusion, that the relation of being an element of any existing[105] distributive class (i) is reflexive, (ii) is transitive, and (iii) takes place between any two individuals.

Finally, we devoted some time to rather historical investigations on existential presuppositions of Syllogistic. There are some traces that suggest to us that Syllogistic required no kind of existential assumptions. As should be also clear, among all mentioned systems of Syllogistic, A-S and O-S remain sound on an empty domain. The Syllogistic which differs most from F-S is V-S, which accepts at least four rules which do not hold in an empty domain. It is, however, worth mentioning that the rules upon which all Syllogistic agreed (i.e. conversion, square of opposites, and syllogisms) hold firmly in an empty domain in K-Interpretation. It is also possible to approach Syllogistic with formal semantics "in hands", and define its key notions.

[105]Clearly, it would be difficult to be a member of non-existing class. We add the phrase 'existing' to indicate that these claims refer to existing individuals or, respectively, classes.

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