On the singularities of non-linear ODEs

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Plan of the Talk:

- Linear special functions
- Nonlinear special functions
- Singularities of ODEs
- Painlevé property and Painlevé transcendents
- Quasi-Painlevé property
- Emden-Fowler equation
Linear Special Functions

- Functions defined by linear ordinary differential equations (ODEs) which have many applications in analysis, number theory, mathematical physics and other fields.

- Example. Hypergeometric equation

\[
\frac{d^2 y(z)}{dz^2} + \left( \frac{c}{z} + \frac{a + b - c + 1}{z - 1} \right) \frac{dy(z)}{dz} + \frac{a b}{z (z - 1)} y(z) = 0,
\]

where \( a, b, c \in \mathbb{C} \), \( y(z) : \mathbb{C} \to \mathbb{C} \).

Gauss hypergeometric series defined by

\[
\,\!_2F_1(a, b, c)(z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,
\]

where \( (a)_n = a(a + 1) \ldots (a + n - 1) \), \( n > 0 \), \( (a)_0 = 1 \), is a solution of the hypergeometric equation.

Singular points of the equation (and, hence, of the solutions since equation is linear) are \( z = 0, 1, \infty \).
There is an integral representation of the solutions of the hypergeometric equation which allows one to calculate a monodromy group (linear representation of the fundamental group of $\mathbb{CP}^1 - \{\text{singular points}\}$ summarizing all analytic continuations of the multi-valued solutions of the equation along closed loops).

- **Example.** Heun equation

\[
\frac{d^2 y(x)}{dx^2} + \left(\frac{c}{x} + \frac{d}{x-1} + \frac{a + b - c - d + 1}{x-t}\right)\frac{dy(x)}{dx} + \frac{abx - q}{x(x-1)(x-t)}y(x) = 0.
\]

Four singularities in the complex plane $x = 0, 1, t, \infty$. Parameter $q$ is called an accessory parameter (in contrast to the hypergeometric equation, it cannot be determined if the monodromy data are given).

Many open questions (e.g., monodromy group, integral representation of solutions). Intriguing applications (e.g., Riemann’s zeta function).
Classification of at least 1 free parameter transformations between \( \binom{2}{1}F_1(a, b, c)(z) \) and Heun function \( Hn(t, q; a, b, c, d)(x) \) [RV-GF], e.g.,

\[
Hn \left( 9, q_1; 3a, 2a + b, a + b + \frac{1}{3}, 2a - 2b + 1 \right)(x) = (1 - x)^{-2a} \binom{2}{1}F_1 \left( a, b, a + b + \frac{1}{3} \right)(z_1),
\]

\[
Hn \left( \frac{8}{9}, q_2; 3a, 2a + b, 2a + 2b - \frac{1}{3}, a + b + \frac{1}{3} \right)(x) = \left( 1 - \frac{9x}{8} \right)^{-2a} \binom{2}{1}F_1 \left( a, b, a + b + \frac{1}{3} \right)(z_2),
\]

where \( q_1 = 18a^2 - 9ab + 6a, \ q_2 = 4a^2 + 4ab - 2a/3, \)

\[
z_1 = -\frac{x(x - 9)^2}{27(x - 1)^2}, \quad z_2 = \frac{27x^2(x - 1)}{(8 - 9x)^2}.
\]

In general, there are about 50 such transformations. Functions \( z_j \) are Belyi functions (branched over 3 points). 38 transformations are related to invariants of elliptic surfaces with 4 singular fibers.

Integral transformations between Heun functions (e.g., Euler type integral transformations found by Slavyanov, GF).
Nonlinear Special Functions

• Riccati equation $y' = a(z)y^2 + b(z)y + c(z)$. Linearizable. Solutions have poles in $\mathbb{C}$, hence, meromorphic functions.

• Elliptic function $\wp(z)$ is a solution of $y'' = p_3(y)$, where $p$ is a degree 3 polynomial. First order nonlinear equation; $\wp(z)$ has poles in $\mathbb{C}$.

• How does one define nonlinear special functions?

In order to define and discuss nonlinear special functions, let us consider different types of singularities of solutions to nonlinear equations.
Singularities of ODEs

A point \( z = z_0 \in \mathbb{C} \) of function \( y(z) : \mathbb{C} \to \mathbb{C} \) is a pole of order \( p \) if

\[
y(z) = \sum_{i=-p}^{\infty} \alpha_i (z - z_0)^i, \quad p \in \mathbb{N}.
\]

A point \( z = z_0 \) of function \( y(z) \) is an algebraic branch point/singularity if

\[
y(z) = \sum_{i=-p}^{\infty} \alpha_i (z - z_0)^{i/n}, \quad p, n \in \mathbb{N}.
\]

If \( n = 1 \), then it is a pole.

**Example 1.** Initial value problem

\[
y' = \frac{1}{2(z + 1)} (y - y^3), \quad y(0) = c,
\]

has a unique solution in a neighborhood of \( z = 0 \) given by

\[
y(z) = c\left[\frac{(1 + z)/(1 + c^2z)}{1 + c^2z}\right]^{1/2}.
\]

Branch points are \( z = -c^{-2} \) (movable as location varies with initial condition) and \( z = -1 \) (fixed singularity).
Roughly, differential equation possesses the Painlevé property if solutions have only movable poles in $\mathbb{C}$ and quasi-Painlevé property if solutions have algebraic branch points.

**Example 2.** The only singularities in $\mathbb{C}$ of solutions of the first and second Painlevé equations

\[
y'' = 6y^2 + z, \quad (P_I)
\]

\[
y'' = 2y^3 + zy + \alpha, \quad \alpha \in \mathbb{C} \quad (P_{II})
\]

are movable poles. Solutions are free from movable branch points (equation admits the Painlevé property). Solutions are meromorphic functions in $\mathbb{C}$.

**Example 3.** Solutions of equation

\[
y'' = 6y^2 + z^2
\]

possess logarithmic singularities

\[
y(z) = \frac{1}{(z - z_0)^2} + \ldots + (\alpha + \frac{1}{7}\log(z - z_0))(z - z_0)^4 + \ldots, \quad \alpha \in \mathbb{C}.
\]

So, multivaluedness of solutions.
Generally, logarithmic psi-series/pseudo-series is of the form:

\[ y(z) = \sum_{m,n=0}^{\infty} a_{m,n} z^p m (\log z)^n. \]

**Example 4.** For Shimomura’s family

\[ y'' = \frac{2(2k + 1)}{(2k - 1)^2} y^{2k} + z, \quad k \in \mathbb{N} \]

solutions have movable algebraic singularities after analytic continuation along a finite length curve.

When \( k = 1 \) we get \( P_I \).

**Classification of singularities:**

- **Fixed vs. movable singularities**

  \[ \frac{d^n y}{d z^n} + p_{n-1}(z) \frac{d^{n-1} y}{d z^{n-1}} + \ldots + p_1(z) \frac{dy}{dz} + p_0(z) y = 0 \]

  Singular points of solutions of linear ODE can be located only at singularities of the coefficients, so singularities are **fixed**.

  Nonlinear equations in general have movable singularities.
Example. Nonlinear equation

\[ \frac{dy}{dz} + y^2 = 0 \]

has general solution

\[ y(z) = (z - z_0)^{-1}, \]

where \( z_0 \) is a constant of integration and a location of singularity. So, location of singularities depends on initial conditions.

• Movable poles/algebraic branch points/logarithmic branch points/more complicated singularities [Ince, Hille, Ablowitz&Clarkson, Golubev, etc].

• Movable algebraic branch point

\[ y' + y^3 = 0, \quad y(z) = (2(z - z_0))^{-1/2} \]

• Movable logarithmic branch point

\[ yy'' - y' + 1 = 0, \quad y(z) = (z - z_0) \ln(z - z_0) + \alpha(z - z_0) \]

• Transcendental singular point \((a \in \mathbb{C} - \mathbb{Q})\)

\[ ayy'' + (1 - a)(y')^2 = 0, \quad y(z) = \alpha(z - z_0)^a \]
• Movable isolated essential singularity

\[ \left( y \frac{d^2y}{dz^2} - \left( \frac{dy}{dz} \right)^2 \right)^2 + 4z \left( \frac{dy}{dz} \right)^3 = 0, \quad y(z) = \alpha \exp\{(z - z_0)^{-1}\} \]

• Non-isolated movable essential singularity

\[ y'' = \frac{2y - 1}{y^2 + 1} (y')^2, \quad y(z) = \tan \{\ln(\alpha(z - z_0))\} \]

• And more complicated.

Since there are so many types of singularities that might occur in solutions of nonlinear ODEs, one might ask: what are the ODEs with solutions possessing simple singularities, e.g., movable poles or algebraic branch points?

Situation with 1st order ODEs is, in a sense, trivial:

Painlevé (1888) proved that for the first order ODEs of the form

\[ G\left( \frac{dy}{dz}, y, z \right) = 0, \]

where \( G \) is a polynomial in \( dy/dz \) and \( y \) with analytic in \( z \) coefficients, the movable singularities of the solutions are poles and/or algebraic branch points (quasi-Painlevé property).
• Works of Painlevé, Picard, Fuchs, Gambier, Bureau: Which equations of type
\[ \frac{d^2y}{dz^2} = F \left( \frac{dy}{dz}, y, z \right), \]
where \( F \) is a rational function of \( dy/dz \) and \( y \) and an analytic function of \( z \), have the \textit{Painlevé property}: solutions have no movable critical points?

• 50 types of equations solutions of which have only movable poles. 44 equations are integrable in terms of linear equations and elliptic functions or reducible to other six equations—\textit{Painlevé equations}:

\begin{align*}
y'' &= 6y^2 + z, \quad (P_I) \\
y'' &= 2y^3 + zy + \alpha, \quad \alpha \in \mathbb{C} \quad (P_{II})
\end{align*}

\[ \frac{d^2y}{dt^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left( \frac{dy}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} + \]
\[ + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left( \alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right), \quad (P_{VI}) \]
\( \alpha, \beta, \gamma, \delta \) being arbitrary parameters.

• First tedious calculations to find a set of \textit{necessary} conditions for the absence of movable critical points.
Next prove that conditions are **sufficient** (difficult problem, settled only recently by Laine, Shimomura, Steinmetz and others, 1990th; also earlier by Hukuhara in unpublished notes).

So, it took almost 100 years to rigorously prove that 6 functions (Painlevé transcendents), solutions of 2nd order nonlinear ODEs, are actually meromorphic functions, i.e., possess the Painlevé property.

Open problem with higher order equations (situation becomes even more complicated as there are natural barriers for analytic continuation of the solutions, e.g., Chazy equation).

The solutions of six Painlevé equations (Painlevé transcendents \((P_I)\)–\((P_{VI})\)) are **nonlinear special functions**. They appear in many areas of modern mathematics (mathematical physics, random matrices, enumerative algebraic geometry, Frobenius manifolds, reductions of integrable PDEs, etc). They are already included in the chapter in the recent update of Abramovich-Stegun’s book *Handbook of Mathematical Functions* in the DLMS project.

It is interesting to note that there is a relation of linear and nonlinear special functions. E.g., elliptic asymptotics of Painlevé transcendents, special solutions for special
values of the parameters expressed in terms of the linear special functions, nice determinant representations of solutions, etc. For instance, \((P_{VI})\) can be regarded as a nonlinear analogue of the hypergeometric equation.

- Painlevé transcendents \((P_{II})\)–\((P_{VI})\) possess Bäcklund transformations (nonlinear recurrence relations, which map solutions of a given Painlevé equation to the solutions of the same/other Painlevé equation but with different values of the parameters and admit an affine Weyl group formulation) [Okamoto, Mazzocco, GF];
- \((P_{I})\)–\((P_{VI})\) admit Hamiltonian formulation, bilinear form. They are irreducible to classical special functions proved recently by Umemura using the differential Galois theory.
- Method of isomonodromy deformations (to study asymptotics and connection formulae): the Painlevé equations (and their multivariable generalizations Garnier and Schlesinger systems) are expressed as a compatibility condition
  \[
  \frac{\partial}{\partial t_j} \frac{dY}{dx} = \frac{d}{dx} \frac{\partial Y}{\partial t_j},
  \]
  of two linear systems of equations
  \[
  \frac{dY}{dx} = AY, \quad \frac{\partial Y}{\partial t_j} = BY.
  \]
Quasi-Painlevé property [GF-RH]

Main question. Under what conditions on $a_j(z)$ do equations of the form

$$y'' = R(z, y, y')$$

have only movable algebraic type singularities at $z = z_0$ after analytic continuation along finite-length curves? We assume that coefficients are analytic at $z = z_0$.

Equations of the form

$$y'' = E(z, y)y'^2 + F(z, y)y' + G(z, y)$$

with certain rational functions $E$, $F$ and $G$ of $y$. This equation includes all the Painlevé equations as particular cases.

We also considered separately

$$y'' = P_1(z, y) := a_n(z)y^n + a_{n-1}(z)y^{n-1} + \ldots + a_0(z)$$

and

$$y'' = P_{n+1}(z, y)y' + P_{n+1}^2(z, y).$$

We prove that solutions of equations above have algebraic branch points after analytic continuation along finite length curves. Coefficients are subject to certain explicit recurrence relations.
Some Remarks on the Proof

- Generalization of Shimomura’s arguments for a particular family of equations
- Very technical proof and many cases to consider depending on \( \lim \inf_{\gamma \ni z \to z_0} |y| \).
- Use of Cauchy theorem and Painlevé lemma
- Construction of certain auxiliary function \( W(z) \) with good properties (such that it is bounded when \( y(z) \) is bounded away from zero) and derivation of resonance conditions with its help.
- Construction of a system for new variables \( (u(z), v(z)) \) with good properties to apply Cauchy theorem.
For example, for equation of the form
\[ y'' = a_N(z)y^N + a_{N-1}(z)y^{N-1} + \ldots + a_0(z), \quad N \geq 2 \]
with
\[ a_N = \frac{2(N + 1)}{(N - 1)^2}, \quad a_{N-1} = 0 \]
one gets the following result:

**Theorem.** Equation admits the quasi-Painlevé property (solutions have algebraic branch points after analytic continuation along a finite length curve) if the following conditions are satisfied:

\[ a''_{N-2}(z) = 0 \]

and when \( N = 2k + 1 \), additionally,

\[ \frac{2}{N - 1}a'_{(N-3)/2}(z) - \frac{1}{2} \sum_{m=1}^{(N-1)/2} \frac{N + 1 - 2m}{N + 1 + 2m} b_m(z)a_{(N-1+2m)/2}(z) = 0 \]

with
\[ \frac{N + 1 - 2n}{(N - 1)^2} b_n(z) = \frac{1}{N - n}a'_{N-n-1}(z) - \frac{1}{2} \sum_{m=1}^{n-1} \frac{N - n - m + 1}{N - n + m + 1} b_m(z)a_{N+m-n}(z), \]

where \( n = 1, \ldots, N - 1 \).
Formal series solution of the form

\[ y(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^{(j-2)/(N-1)} \]

with \( c_{2(N+1)} \) is arbitrary is convergent (and there are 2 different leading order behaviors depending on whether \( N \) is even or odd).

Let \( y(z) \) be a solution that can be continued analytically along a curve \( C \) up to but not including the endpoint \( a \), where \( a_j(z) \)'s are analytic on \( C \). If \( C \) is of finite length, then \( y \) has a convergent series expansion about \( a \) of the form above.

The function \( W(z) \) given by

\[ W(z) = y'(z)^2 + \left( \sum_{k=1}^{N-1} b_k(z)/y^k(z) \right) y'(z) - 2 \sum_{k=1}^{N+1} a_{k-1}(z)y^k(z)/k, \]

is bounded when \( y(z)^{-1} \) is bounded.

**Examples.** When \( N = 2 \) and \( N = 3 \), the above conditions on the coefficients distinguish \((P_I)\) and \((P_{II})\) respectively.

\[ N = 2 : \quad a''_0(z) = 0 \]

\[ N = 3 : \quad -2a'_0(z) + c_0a''_1(z) = 0, \quad c_0^2 = 1. \]

So, we manage the general class of equations simultaneously.
Open questions

• General second order equations

\[ F \left( \frac{d^2 y}{dz^2}, \frac{dy}{dz}, y, z \right) = 0. \]

• Higher order equations

\[ F \left( \frac{d^n y}{dz^n}, \ldots, \frac{dy}{dz}, y, z \right) = 0, \ n > 2. \]

• Computationally difficult problem and also serious complications: third order equations may have movable natural boundaries (a line of singularities through which the solution cannot be analytically continued). Chazy equation: \( y''' = 2yy'' - 3y'^2 \).

• Smith example

\[ y'' + 4y^3y' + y = 0 : \]

accumulation of algebraic singularities along infinite length curves.

By accumulation point it is meant that for any \( \epsilon > 0 \) there exists a straight line segment \( l \) in the disk of radius \( \epsilon \) centered at \( z_0 \) with endpoints \( z_1 \in \gamma \) and \( z_2 \), where \( \gamma \) is an infinite length curve ending at \( z_0 \), such that analytic continuation of the solution along \( \gamma \) up to \( z_1 \) and then along \( l \) ends in an algebraic singularity at \( z_2 \).
The singularities of the Emden-Fowler type equations

[R. Kycia, GF, On the singularities of the Emden-Fowler type equations, to appear.]

The Emden-Fowler equation

\[
\frac{d^2 u(x)}{dx^2} + \frac{\alpha}{x} \frac{du(x)}{dx} + \delta x^n u(x)^p = 0
\] (1)

has many application in physics. We assume that \(\alpha > 0\) and \(\delta \neq 0\) are real parameters, \(n\) is an integer such that \(n > -2\) and \(p > 1\) is a natural number.

There exists a power series solution at the origin which is convergent in a finite circle centred at the origin in the complex plane because of the existence of singularities on the boundary of this circle. These singularities move when we change the initial data and/or the parameters of the equation.

It is straightforward to show that equation (1) has a local analytic solution near \(x = 0\) of the form

\[
u(x) = \sum_{k=0}^{\infty} a_k x^k.
\] (2)

One can prove the existence of movable singularities of the solution (2) of equation (1). This result can be regarded as a generalization of the results of Hunter for the Lane-Emden equation. In particular, a nonzero analytic solution (2) of equation (1) has \(n + 2\)
singularities located symmetrically with respect to the origin on the rays connecting the origin with all \((n + 2)\) roots of \(-1\) in the complex plane (see, for instance, Fig. 1 drawn by R. Kycia).
Summary

- Linear special functions, defined by linear ODEs, appear in many areas of mathematics. They have nontrivial transformations between themselves.

- Solutions of nonlinear ODEs have complicated singularities. Solutions of Painlevé equations (2nd order equations) have only movable poles and appear in many areas of mathematics, so they are nonlinear special functions. There are very few results on higher-order equations or multivariable generalizations.

Thank you very much for your attention!