Introduction to elliptic functions
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Graduate lecture course

Description. Starting from the works of Gauss, Cauchy, Abel, Jacobi, Eisenstein, Riemann, Weierstrass, Klein and Poincare, the theory of meromorphic functions of a complex variable has developed significantly and has direct links with analysis, differential equations, algebra, number theory, potential theory, geometry and topology. It makes an interesting and important topic for study. Since the theory is very rich, we shall mainly concentrate on the analytic viewpoint in the course. Connections with other areas will be discussed as well. We shall start with comprehensive introduction to elliptic functions, which are doubly periodic functions. They arose from attempts to evaluate certain integrals associated with the formula for the circumference of an ellipse. They can be regarded as the meromorphic functions of on the torus. Moreover, the elliptic functions are the rational functions of the Weierstrass function and its derivative, these two functions being related by a first order nonlinear ordinary differential equation. Other topics covered include automorohic functions, applications to number theory. If time permits, other special functions like hypergeometric functions and their confluences, Lame functions etc will be discussed.

Prerequisite courses: ordinary differential equations, analytic functions.

Suitable for graduate students and also for 3-4th year students (30+30 hours of lectures and problem classes).

Lectures:

1. Functions sin, sh, sl, sn; their properties. Connection to the curves in the plane and to elliptic integrals. [1]
2. The generalized sine in the complex plane, main properties. [1]
3. Weierstrass approach to elliptic functions. [5]
4. General properties of elliptic functions. [5]
5. Derivation of an ODE for the Weierstrass function (two methods). Addition theorem for the Weierstrass function. [5]
6. Expression of arbitrary elliptic functions in terms of the Weierstrass elliptic function (and its derivative). [5]
7. Legendre's relation. Sigma function. [5]
8. Expressions of elliptic functions in terms of sigma functions. Weierstrass function and elliptic function as functions of the periods. [5]
9. Riemann surfaces, genus, holomorphic differentials. [13, 14, 15]
10. Introduction to theta functions. [5, 10, 20, 6]

11. Cubic curves: addition, singular points, normal forms, rational parametrization of singular curves. [16]

12. Rational points on elliptic curves. Points of finite order and Nagell-Lutz theorem. Mordell's theorem with a sketch of the proof. [17, 24]

13. Modular functions. [19, 5]

14. Congruence subgroups. Modular forms. [19]

15. Configuration spaces of points. Moduli spaces of elliptic curves. Invariants. The hypergeometric equation. [22]

16. Lame functions. [21, 23]

Problem classes and seminars:

1. Elliptic integrals of the first, second and third kind (Legendre's canonical form). Series expansions. Landen's transform. Integration along the closed path in the complex plane. [2]

2. Euler's method to derive addition theorem for the generalized sine [1]. Legendre's theorem [6].

3. Lecture 4

4. Connection between the Jacobi and Weierstrass elliptic functions. Examples of integration of Weierstrass functions. Degeneration of elliptic functions. Examples of differential equations which can be integrated with the help of the Weierstrass function. [2]

5. Jacobi's sn function in the complex domain. Determinantal form of addition theorem for the Weierstrass elliptic function. [8]

6. Conformal maps using the elliptic functions. [2] Uniformization of elliptic curves. [8]

7. Calculations of genus of the Riemann surface using the Riemann-Roch formula. [13, 14]

8. Expression of $K(k)$ in terms of the hypergeometric function. Euler characteristics and genus. [14, 15] Continuation of conformal maps using elliptic functions. [2]

9. Lecture 10

10. Periods of the Hesse normal form and the Heun's differential equation. [18]

11. Lecture 12

12. Lecture 13
13. Lecture 14
14. Periods of elliptic integrals. [5]

Exam topics:
Lectures L1, L2, L4, L6, L7, L8, L9, L11, L13.

References

[1] A. I. Markushevich. Zamechatel'nye sinusy. Nauka, Moskva, 1974 (in Russian).

[2] A. M. Zhuravskij. Spravochnik po ellipticheskim funkcijam. Izd. Akad. Nauk SSSR, Moskva, 1941 (in Russian).

[3] F. Leja. Funkcje analityczne i harmoniczne. Monografie Matematyczne 29, PWN, Warszawa-Wrocław, 1952.

[4] S. Saks, A. Zygmund. Funkcje analityczne. Monografie Matematyczne 28, PWN, Warszawa 1952.

[5] A. Hurwitz, R. Courant. Vorlesungen Über Allgemeine Funktionentheorie Und Elliptische Funktionen. Nabu Press, 2010.

[6] R. Wong. Special functions. Cambridge Studies in Adv. Math. 126, CUP, 2010.

[7] J. V. Armitage, W.F. Eberlein. Elliptic functions. LMS Students Texts 67, CUP, 2006.

[8] E. T. Whittaker, G. N. Watson. A Course of Modern Analysis. CUP, 1927.

[9] N. I. Akhiezer. Elements of the Theory of Elliptic Functions. Transl. AMS 79 (English verstion), Nauka, Moscow, 1970 (Russian version).

[10] D. Lawden. Elliptic functions and applications. Appl. Math. Sci. 80, Springer, 1989.

[11] K. Chandrasekharan. Elliptic functions. Springer-Verlag, Berlin-Heidelberg, 1985.

[12] P. du Val. Elliptic functions and elliptic curves. LMS Lecture Notes Series 9, CUP, 1973.

[13] G. Jones, D. Singermann. Complex functions: an algebraic and geometric viewpoint. CUP, 1987.

[14] M. Huttner. Surfaces de Riemann. Cours de DEA 1996-1997, University of Lille.

[15] Lecture notes of H. Zoladek on Monodromy group, University of Warsaw, 2011.

[16] V. Prasolov, Yu. Solovyev. Elliptic functions and elliptic integrals. Transl. Math. Monographs, AMS, 1997.

[17] J. H. Silvermann, J. Tate. Rational points on elliptic curves. Undergraduate Texts in Math., Springer. 1992.

[18] K. Matsumoto. A Heun differential equation derived from the Gauss hypergeometric differential equation. preprint, 2005.

[19] T. Apostol. Modular functions and Dirichlet series in number theory. Graduate Texts in Mathematics 41, Springer, 1989.

[20] D. Mumford. Tata lectures on Theta I. Modern Birkhäuser Classics, Birkhauser, Boston, 2006.

[21] H. Bateman, A. Erdelyi. Higher transcendental functions I, II, III. Dover Publications, 2007.

[22] M. Yoshida. Hypergeometric functions, my love. Modular interpretations of configuration spaces. Aspects of Mathematics E 32, Vieweg in Braunschweig, 1997.

[23] R. Burger, G. Labahn, M. van Hoeij. Closed form solutions of linear ODEs having elliptic function coefficients. ISSAC 2004, 58-64, ACM, New York, 2004.

[24] J. W. S. Cassels. Lectures on elliptic curves. LMS Students Texts 24, CUP, 1991.

Main textbooks: [5, 8, 9] for elliptic functions, [13, 17] for Riemann surfaces and elliptic curves, [19] for modular functions.