Asymptotic forms and asymptotic expansions of solutions to the Painlevé equations

Irina V. Goryuchkina

Department of Singular Problems,
Keldysh Institute of Applied Mathematics, Moscow

August 8 - August 13, 2011
1. ODE’s under consideration and types of their solutions

2. On the sixth Painlevé equation

3. On results obtained for P6

4. Examples of five types of solutions to P6

5. On convergent formal solutions

6. Boutroux type asymptotic forms
We study ordinary differential equations (ODE’s) that can be reduced to the form

\[ f(x, y, y', y'', \ldots, y^{(n)}) = 0, \]  

where \( x \) and \( y \) are independent and dependent variables correspondingly, \( y' = dy/dx \), \( f \) is a polynomial in its variables \( x, y, y', y'', \ldots, y^{(n)} \).

We can find all asymptotic expansions of solutions to the equation (1) near its singular and nonsingular points of the six types.

Namely, power, power-logarithmic, complicated, exotic, semiexotic and exponential.
Types of asymptotic expansions of solutions

At $x = 0$ we seek all asymptotic expansions of solutions to the equation (1) of the form

$$y = c_r x^r + \sum_s c_s x^s,$$

where power exponents $r$ and $s$ are complex, $\Re s > \Re r$, $\Re s$ increase. They belong to one of three types: power, power-logarithmic or complicated. Expansions of these types have finite number of power exponents $s$ with the same $\Re s$.

Coefficients $c_r$ and $c_s$ are varied according to types:

**Type 1.** $c_r$ and $c_s$ are constants (power expansions);

**Type 2.** $c_r$ is constant, $c_s$ are polynomials in $\log x$ (power-logarithmic expansions);

**Type 3.** $c_r$ and $c_s$ are series in decreasing powers of $\log x$ (complicated expansions).
At $x = 0$ we also consider expansions of the form

$$y = \sum_{\rho} c_\rho x^\rho + \sum_{s} c_s x^s,$$

where power exponents $\rho$ and $s$ are complex, all $\text{Re } \rho$ are the same, $\text{Re } \rho < \text{Re } s$, $\text{Re } s$ increase, first sum contains more than one term, complex coefficients $c_\rho$ and $c_s$ are polynomials in $\log x$.

We also suppose $\arg x$ bounded.

We differ two types of the expansions (3).

**Type 4.** The first sum in (3) contains infinite number of terms, but $\text{Im } \rho$ are bounded either below or above. Coefficient $c_\rho$ is constant for extreme value of $\text{Im } \rho$ (exotic expansions).
Types of asymptotic expansions of solutions

**Type 5.** The first sum in

\[ y = \sum_{\rho} c_{\rho} x^\rho + \sum_{s} c_{s} x^s, \]  

contains a finite number of terms. Coefficients \( c_{\rho} \) are constants for both extreme values of \( \text{Im}\rho \). The numbers of power exponents \( s \) with the same real parts \( \text{Re} s \) are finite (semiexotic expansions);

Let \( \min(\text{Im}\rho) \) in (3) be reached at \( \rho = \rho_1 \), and \( \max(\text{Im}\rho) \) at \( \rho = \rho_2 \). A semiexotic expansion (3) has two inverse expansions

\[ (y^{-1})_1 = c_{\rho_1}^{-1} x^{-\rho_1} \left( \sum_{\sigma} b_{\sigma} x^\sigma + \sum_{t} b_{t} x^t \right), \]  

\[ (y^{-1})_2 = c_{\rho_2}^{-1} x^{-\rho_2} \left( \sum_{\sigma} \tilde{b}_{\sigma} x^\sigma + \sum_{t} \tilde{b}_{t} x^t \right), \]

where \( \text{Re} \sigma = 0 \), which are exotic ones.
Types of asymptotic expansions of solutions

**Type 6.** In addition, there are exponential expansions

\[ y = \sum_{k=0}^{\infty} b_k(x) C^k e^{k\phi(x)}, \]

(6)

where \( b_k(x) \) and \( \phi(x) \) are power series, but \( C \) is arbitrary constant.

All Painlevé equations (P1 – P6 equations) can be reduced to the form

\[ f(x, y, y', y'') = 0, \]

where \( f \) is a polynomial of its variables. So all expansions of their solutions of the six types can be found.
The sixth Painlevé (P6) equation has form

$$y'' = \frac{(y')^2}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) - y' \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right)$$

$$+ \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[ a + b \frac{x}{y^2} + c \frac{x-1}{(y-1)^2} + d \frac{x(x-1)}{(y-x)^2} \right],$$

where $a, b, c, d$ are complex parameters, $x$ and $y$ are complex variables independent and dependent correspondingly. It has three singular points $x = 0, 1, \infty$.

Near its three singular points $x = 0, 1, \infty$ and near its nonsingular points $x = x_0 \neq 0, 1, \infty$ for all values its four complex parameters $a, b, c, d$ we found all asymptotic expansions of its solutions of the five types:

- power,
- power-logarithmic,
- complicated,
- exotic,
- semiexotic.

Exponential expansions are absent.
On the sixth Painlevé equation

The equation (7) has three main symmetries that allow to transfer solutions near of one its singular point to solutions near the same or another its singular point. These symmetries are associated with three changes of variables

1. \( x = z, \ y = z/w; \)
2. \( x = 1/z, \ y = 1/w; \) (8)
3. \( x = 1 - z, \ y = 1 - w. \)

The first symmetry takes the expansions of solutions in a neighborhood of zero to the expansions of solutions in a neighborhood of infinity (and vice versa).

The second symmetry preserves the position of singular points.

The third symmetry takes the expansions of solutions in a neighborhood of zero to the expansions of solutions in a neighborhood of unity.
First we obtain asymptotic expansions of such solutions to the equation (7) near the singular point $x = 0$, for which the order of the first term is less than unity. We refer to these expansions as basic expansions. They form 21 families.

We obtain all other asymptotic expansions of solutions to the equation (7) near its three singular points $x = 0$, $x = 1$, and $x = \infty$ from the basic expansions using symmetries (8) of the equation. Altogether they form 117 families [1].

Studies of asymptotic forms and asymptotic expansions of solutions to P6 were performed by authors: *S. Shimomura, M. Jimbo, H. Kimura, K. Okamoto, I. V. Gromak, B. A. Dubrovin, M. Mazzocco, D. Guzzetti*, and many others.
We obtain all asymptotic expansions of solutions to the sixth Painlevé equation near all its three singular points $x = 0$, $x = 1$, and $x = \infty$ for all values of four complex parameters $a, b, c, d$ of this equation [1]. They belong to expansions of five types: power, power-logarithmic, complicated, semiexotic, and exotic. They form 117 families. Among them 33 families of expansions are power, 12 families are power-logarithmic, 18 families are complicated, 48 families are exotic, 6 families are semiexotic. The most of them are new.

In neighborhood of a regular point $x = x_0 \neq 0, 1, \infty$ of the sixth Painlevé equation (7) for all values its four parameters there exist 17 families of power expansions of its solutions [2]. They are Laurent or Taylor series. Among them 1 family of expansions has the pole of the second order, 2 families of expansions have poles of the first order, other ones are families of Taylor expansions. 8 families of these expansions are new.
For $x \to 0$ there exists family of **power expansions** of solutions to the equation (7) with two parameters $c_r$ and $r$, and constant coefficients, which has form

$$
\mathcal{A}_0 : y = c_r x^r + \sum_s c_s x^s,
$$

(9)

where complex power exponent $r$ is arbitrary with $\text{Re} r \in (0, 1)$, complex power exponents $s \in \{r + lr + m(1-r), l, m \geq 0; l + m > 0; l, m \in \mathbb{Z}\}$; complex coefficient $c_r$ is arbitrary nonzero constant, other complex coefficients $c_s$ are uniquely determined constants.

The family $\mathcal{A}_0$ was known. Family $\mathcal{A}_0$ exists for all values of parameters of equation (7).

Altogether there are 33 families of power expansions of solutions to P6.
Example of power-logarithmic expansions of solutions to P6

For $x \to 0$ and $a = c \neq 0$, $2\sqrt{2}a \in \mathbb{Z}\setminus\{0\}$ there exists family of power-logarithmic expansions of solutions to the equation (7) of the form

$$B_2 : \quad y = 2 + \sum_{s=1}^{\infty} c_s(\log x)x^s,$$  \hspace{1cm} (10)

where coefficients $c_s(\log x)$ are polynomials in $\log x$.
Altogether there are 12 families of power-logarithmic expansions of solutions to P6.
Example of complicated expansions of solutions to P6

For $x \to 0$ and $a \neq c \neq 0$ there exists family of complicated expansions of solutions to the equation (7) of the form

$$\mathcal{B}_3: \quad y = \psi_0 + \sum_{\sigma=1}^{\infty} \psi_\sigma x^\sigma,$$

where

$$\psi_0 = \frac{2}{c-a} \frac{1}{\log^2 x} + \frac{c_{-3}}{\log^3 x} + \sum_{s=4}^{\infty} \frac{c_{-s}}{\log^s x} = \frac{2(c-a)}{(c-a)^2(\log x + C_0)^2 - 2a},$$

coefficients $c_{-3}$ and $C_0$ are arbitrary constants, other coefficients $c_{-s}$ are uniquely determined constants; $\psi_\sigma$ are series in decreasing powers of $\log x$.

Altogether there are 18 families of complicated expansions of solutions to P6.
Example of exotic expansions of solutions to P6

For $x \to 0$ there exists family of exotic expansions of solutions to the equation (7) with two parameters $C_1$ and $\rho$, and constant coefficients, which has form

$$\mathcal{B}_0^\tau : y = \frac{\rho^2}{\beta \cos^2[\log(C_1x)\gamma] + \alpha \sin^2[\log(C_1x)\gamma]} + \sum_{\text{Res} \geq 1} c_s x^s = x^\rho \left( c_{\rho} + \sum_{k=1}^{\infty} \tilde{c}_k x^{k\rho} \right) + \sum_{\text{Res} \geq 1} c_s x^s,$$

(12)

where complex power exponent $\rho$ is pure imaginary nonzero arbitrary constant, $s \in \{\rho + l\rho + m(1 - \rho); l, m \geq 0; l + m > 0; l, m \in \mathbb{Z}\}$, $\tau = \text{sgn}(\text{Im} \rho)$, $\alpha + \beta = (\rho^2 - 2c + 2a)/(2a)$, $\alpha \beta = \rho^2/(2a)$, $2\gamma = i\rho$, complex coefficients $c_{\rho}$ and $C_1$ are nonzero arbitrary constants and interrelated, other complex coefficients $\tilde{c}_k$ and $c_s$ are uniquely determined constants. Altogether there are 48 families of exotic expansions of solutions to P6.
Example of semiexotic expansions of solutions to P6

For $x \to 0$ there exists a family of semiexotic expansions to solutions of the equation (7) with two parameters $c_\rho$ and $\rho$, and constant coefficients, which has form

$$H_0 : \quad y = c_\rho x^\rho + c_1 x + c_{2-\rho} x^{2-\rho} + \sum_s c_s x^s + \ldots,$$

where power exponent $\rho$ is complex, $\rho - 1$ is pure imaginary arbitrary constant, $s$ runs the set

$$\{l + k(\rho - 1); l, k \in \mathbb{Z}; l \geq 2, |k| \leq l\},$$

complex coefficient $c_\rho$ is arbitrary constant, other complex coefficients $c_1$, $c_{2-\rho}$ and $c_s$ are uniquely determined constants.

Altogether there are 6 families of semiexotic expansions of solutions to P6.
On convergent formal solutions

We consider an ordinary differential equation of the form

\[ f(x, y, y', y'', \ldots, y^{(n)}) = 0, \]  

(1)

where \( x \) and \( y \) are independent and dependent variables correspondingly, \( y' = dy/dx \), \( f \) is a polynomial of its variables \( x, y, y', y'', \ldots, y^{(n)} \).

Let for \( |x| \to 0 \) and \( \arg(x) \in (-\pi, \pi) \), equation (1) have the formal solution

\[ y = \sum c_s x^s, \ s \in K \subset \mathbb{C}, \]  

(8)

where power exponents \( s \) are complex, \( \Re s \) increase, number of power exponents \( s \) with the same real parts \( \Re s \) is finite, coefficients \( c_s \) are complex constants.
By the substitution

$$y = \sum_{s=s_0}^{s_m} c_s x^s + u,$$

(14)

where $m \in \mathbb{Z}$, $m \geq 0$, $\text{Re} s_m \geq n$, we reduce equation (1) to the form

$$f_1(x, u) \overset{\text{def}}{=} \mathcal{L}(x) u + g(x, u, u', \ldots, u^{(n)}) = 0,$$

(15)

where the linear differential operator

$$\mathcal{L}(x) = x^v \sum_{l=1}^{n} a_l x^l \frac{d^l}{dx^l},$$

$\mathcal{L}(x) \not\equiv 0$, $v \in \mathbb{C}$, $a_l$ are complex constants.

The function $g$ contains terms independent of $u, u', \ldots, u^{(n)}$, linear terms in $u, u', \ldots, u^{(n)}$ of the form $c x^{v_1 + l} u^{(l)}$ with $\text{Re} v_1 > \text{Re} v$, $l \leq n$, $c = \text{const} \in \mathbb{C}$, and nonlinear terms in $u, u', \ldots, u^{(n)}$. 
The linear differential operator \( \mathcal{L}(x) \) has eigenvalues \( \lambda_1, \ldots, \lambda_n \). We order these eigenvalues by increasing real parts: \( \text{Re} \lambda_1 \leq \cdots \leq \text{Re} \lambda_n \).

We suppose that \( \text{Re} s_m > \text{Re} \lambda_n \) in substitution (2). Then for \( |x| \to 0 \), \( \arg(x) \in (-\pi, \pi) \) equation (15) has a unique solution of the form

\[
  u = \sum_{s=s_{m+1}}^{\infty} c_s x^s, \quad (16)
\]

where power exponents \( s \in \mathbb{C} \), \( \text{Re} s \) increase, complex coefficients \( c_s \) are uniquely determined constants.

In 2004 Prof. Bruno had formulated the theorem on convergence of a power series solution to an ordinary differential equation.
Theorem (1)

If in the equation

\[ f_1(x, u) \overset{\text{def}}{=} \mathcal{L}(x)u + g(x, u, u', \ldots, u^{(n)}) = 0, \]

which we obtain after substitution

\[ y = \sum_{s=s_0}^{s_m} c_s x^s + u, \]

to the equation

\[ f(x, y, y', y'', \ldots, y^{(n)}) = 0, \]

the order of the highest derivative in \( \mathcal{L}(x)u \) is equal to the order of the highest derivative in the sum \( f_1 \), then the series

\[ u = \sum_{s=s_m+1}^{\infty} c_s x^s, \]

converges for sufficiently small \( |x| \) and \( \arg x \in (-\pi, \pi) \).
In 2010 we have proved Theorem (1) in the case of rational power exponents $s$ of power expansion $y = \sum c_s x^s$. It is published in the article


In 2011 we have proved Theorem (1) in the case of complex but not rational power exponents $s$ of power expansion $y = \sum c_s x^s$. It is published in preprint

According to Theorem (1) all power expansions of solutions to the sixth Painlevé equation near its three singular points are convergent. Near regular point all expansions converge according to Theorem (1), and in some cases according to the Cauchy Theorem.
Existence Boutroux type asymptotic forms to P6

For $|x| \to \infty$ we also can find asymptotic forms of solutions to the equation (1) of the form

$$y = x^\alpha \varphi(x^\beta),$$

where $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, $\varphi(x^\beta)$ is elliptic or periodic function (Boutroux type asymptotic forms).

Theorem (2)

Near singular points $x = 0, 1, \infty$ of the sixth Painlevé equation there is not Boutroux type asymptotic forms of solutions contrary to other Painlevé equations [6], [7], [8].

In our article [6] we found Boutroux asymptotic forms of solutions to P1 and P2. First they were found by Boutroux. In other two articles [7] and [8] we found Boutroux type asymptotic forms of solutions to P3 and P4 correspondingly.


References


Bruno, A.D., Goryuchkina, I.V., *Asymptotic Forms of Solutions to the Third Painlevé Equation*. Doklady Mathem. 2008. 78 (2). 765–768. (in English)

Bruno, A.D., Goryuchkina, I.V., *Asymptotic Forms of Solutions to the Fourth Painlevé Equation* Doklady Mathem. 2008. 78 (3). 868–873. (in English)
THANK YOU FOR ATTENTION!