Default logic (Reiter)

Default:

$$\frac{Bird(x) : \ Flies(x)}{Flies(x)}$$

Default theory:

Axioms + defaults

Definition A *default* is any expression of the form

$$\frac{A(\overline{x}) : \ B_1(\overline{x}), \ldots, B_k(\overline{x})}{C(\overline{x})}$$

where $A(\overline{x}), B_1(\overline{x}), \ldots, B_k(\overline{x})$ and $C(\overline{x})$ are first-order formulas whose free variables are among those of $\overline{x} = (x_1, \ldots, x_n)$.

$A(\overline{x})$ – the *prerequisite*;
$B_1(\overline{x}), \ldots, B_k(\overline{x})$ – the *justifications*;
$C(\overline{x})$ – the *consequent*.

Definition A *default theory* is a pair $T = \langle$W,D$\rangle$, where W is a set of first-order sentences and D is a set of defaults.

A set of theorems derivable from a default theory T is called an *extension of* T.

An extension should include whatever can be inferred from axioms either by ordinary logic or by applying defaults.

If E is an extension of $T = \langle$W,D$\rangle$, then:

(1) E should be deductively closed with respect to standard logic: E=Th(E);

(2) E should contain axioms: W$\subseteq$E;

(3) E should contain a maximal set of conclusions obtainable by applying defaults from D.

Note:

(1)-(3) imply that an extension of a theory $\langle$W,D$\rangle$ is always given by Th(X$\cup$W), where X is a set of conclusions obtainable by applying defaults from D.

Example

W={(1) $Basket - Player(Bill)$};

$$\text{D}=\left\{\frac{Basket - Player(x) : \ Tall(x)}{Tall(x)}, \ \frac{Basket - Player(x) : \ Nimble(x)}{Nimble(x)}\right\}.$$

E=Th($\{(1), Tall(Bill), Nimble(Bill)\}$).

Let T obtain by adding

(2) $\neg Tall(Bill) \vee \neg Nimble(Bill)$.

E1=Th((1),(2),$Tall(Bill)$);

E2=Th((1),(2),$Nimble(Bill)$).

Open defaults and their instances

Definition A default $A(\overline{x}):B_1(\overline{x}),\ldots,B_k(\overline{x})/C(\overline{x})$ is called *open* iff at least one of $A(\overline{x}), B_1(\overline{x}),\ldots,B_k(\overline{x}), C(\overline{x})$ contains a free variable; otherwise it is called *closed*. A default theory is said to be *open* iff it contains at least one open default; otherwise it is said to be *closed*.

An *instance* of an open default is the result of uniformly replacing all free occurrences of its variables by ground terms.

$$\frac{Bird(cousin(Bob)) : \ Flies(cousin(Bob))}{Flies(cousin(Bob))}$$

$$\frac{Bird(x) : \ Flies(x)}{Flies(x)}$$

$$\boxed{T \quad \longrightarrow \quad \text{CLOSED(T)}}$$

Example Let T consist of:

$$Bird(Tweety) \wedge Fish(Nelly)$$

$$\frac{Bird(x) : Has - Wings(x)}{Has - Wings(x)}.$$

CLOSED(T):

$$Bird(Tweety) \wedge Fish(Nelly)$$

$$\frac{Bird(Tweety) : Has - Wings(Tweety)}{Has - Wings(Tweety)}$$

$$\frac{Bird(Nelly) : Has - Wings(Nelly)}{Has - Wings(Nelly)}.$$

Example Let T consist of:

$$Fish(Bob)$$

$$\forall x.Fish(x) \supset Fish(father(x))$$

$$\frac{Fish(x) : Swims(x)}{Swims(x)}.$$

CLOSED(T):

$$Fish(Bob)$$

$$\forall x.Fish(x) \supset Fish(father(x))$$

$$\frac{Fish(Bob) : Swims(Bob)}{Swims(Bob)}$$

$$\frac{Fish(father(Bob)) : Swims(father(Bob))}{Swims(father(Bob))}$$

etc.

Example Let T consist of:

$$\exists x.Butterfly(x)$$

$$\frac{Butterfly(x) : Brightly - Coloured(x)}{Brightly - Coloured(x)}.$$

We would like to infer

$$\exists x.Butterfly(x) \wedge Brightly - Coloured(x).$$

CLOSED(T):

$$Butterfly(a)$$

$$\frac{Butterfly(a) : Brightly - Coloured(a)}{Brightly - Coloured(a)}.$$

Example Let T consist of:

$$Birthday(Bill)$$

$$\frac{: \exists x.Friend(x, Bill)}{\exists x.Friend(x, Bill)}$$

$$\frac{Birthday(y) \wedge Friend(x, y) : Gives - Gift(x, y)}{Gives - Gift(x, y)}$$

The first default is replaced by

$$\frac{: \exists x.Friend(x, Bill)}{Friend(b, Bill)}$$

and then we proceed in the usual way.

Definition A *Skolemized form* of a sentence A is any universal sentence obtained from any prenex normal form of A by the process of Skolemization.

Definition A *Skolemized form* of a default

$$\frac{A(\overline{x}) : \ B_1(\overline{x}), \ldots, B_k(\overline{x})}{C(\overline{x})}$$

obtains by replacing $C(\overline{x})$ by any Skolemized form of $\forall C(\overline{x})$ and deleting all quantifiers from the consequent of the resulting default.

Definition A Skolemized form of a default theory T is any theory obtained by replacing defaults and axioms of T by their Skolemized forms.

Definition Let $T = \langle$W,D$\rangle$ be a default theory. By TERMS(T) we denote the set of all ground terms constructable using function constants occurring in W and D.

Definition Let T be a default theory. A *closure* of T, written CLOSED(T), is any theory obtained by the following construction:

1. If T is closed, then put CLOSED(T)=T. Otherwise:
2. Replace T by its Skolemized form, $T1$.
3. Let $T2$ be the theory obtained from $T1$ by replacing all its open defaults by their instances over TERMS($T1$). This is the desired closure of T.

Definition Let T be a default theory over L_T. E is an extension of T iff E=E'$\cap L_T$ and E' is an extension of CLOSED(T).

Closed default theories

The following conditions can be naturally imposed on an extension E of a theory $\langle$W,D$\rangle$:

(1) E=Th(E);

(2) W$\subseteq$E;

(3) E should contain a maximal set of conclusions obtainable by applying defaults from D.

Notice:

Any conclusion derivable by applying a closed default is just its consequent.

(3') E should contain the consequent of any applicable default from D.

There still remains the problem of specifying a formal criterion of default's applicability.

$$\frac{A:\ B_1,\ldots,B_k}{C}$$

(3'') If $(A : B_1,\ldots,B_k/C) \in$ D, $A \in$ E and $\neg B_1 \notin$ E, $\ldots, \neg B_k \notin$ E, then $C \in$ E.

Closure conditions:

(1) E=Th(E);

(2) W⊆E;

(3'') If $(A : B_1,\ldots,B_k/C) \in$ D, $A \in$ E and $\neg B_1 \notin$ E, $\ldots, \neg B_k \notin$ E, then $C \in$ E.

Closure conditions specify what should be in an extension. However, they say nothing about what should be off.

"Definition" E is an extension of a closed default theory iff E is the smallest set satisfying the closure conditions.

"Definition" E is an extension of a closed default theory iff E is a minimal set satisfying the closure conditions.

$T = \langle \{P(a)\}, \{P(a) : Q(a)/Q(a)\} \rangle$

Both

Th($\{P(a), Q(a)\}$) and Th($\{P(a), \neg Q(a)\}$)

satisfy the closure conditions.

Problem:

Assume that E and Γ(E) denote the same extension of ⟨W,D⟩. How to define Γ(E) by means of E?

Solution:

Let Γ(E) be the smallest set satisfying:

(1) Γ(E)=Th(Γ(E));

(2) W⊆ Γ(E);

(3) If $(A : B_1,\ldots,B_k/C) \in$ D, $A \in$ Γ(E) and $\neg B_1 \notin$ E, $\ldots, \neg B_k \notin$ E, then $C \in$ Γ(E).

Definition Let $T = \langle$W,D$\rangle$ be a closed default theory over L_T. For any set of sentences S$\subseteq L_T$, let Γ(S) be the smallest set of sentences from L_T satisfying:

(1) Γ(S)=Th(Γ(S));

(2) W$\subseteq \Gamma$(S);

(3) If $(A : B_1, \ldots, B_k/C) \in$ D, $A \in \Gamma$(S) and $\neg B_1 \notin$ S, $\ldots, \neg B_k \notin$ S, then $C \in \Gamma$(S).

A set of sentences E$\subseteq L_T$ is an *extension* of T iff E=Γ(E).

Example Reconsider:

$T = \langle\{P(a)\}, \{P(a) : Q(a)/Q(a)\}\rangle$.

T gives rise to two minimal sets satisfying the closure conditions:

E1=Th($\{P(a), Q(a)\}$) and

E2=Th($\{P(a), \neg Q(a)\}$).

Since

Γ(E1)=Th($\{P(a), Q(a)\}$)=E1

Γ(E2)=Th($\{P(a)\}$)$\neq$ E2

only E1 is an extension of T.

Example Let T be a theory with the empty set of axioms and one default

$$\frac{p}{\neg p}.$$

T has no extension. To show it, note that the only candidates are

E1=Th($\{\neg p\}$) and E2=Th($\{\}$).

Since

Γ(E1)=Th($\{\}$)$\neq$E1

Γ(E2)=Th($\{\neg p\}$)$\neq$E2

neither E1 nor E2 is an extension of T.

Theorem If $T = \langle$W,D$\rangle$ is a closed default theory, then a set E of sentences is an extension of T iff E=$\bigcup_{i=0}^{\infty}$ E$_i$, where

E$_0$=W, and for i$\geq$ 0

E$_{i+1}$=Th(E$_i$) $\cup$ $\{C : (A : B_1, \ldots, B_k/C) \in$ D, where E$_i \vdash A$ and $\neg B_1, \ldots, \neg B_k \notin$ E$\}$.

Corollary A closed default theory $\langle$W,D$\rangle$ has an inconsistent extension iff W is inconsistent.

Theorem If E and F are extensions of a closed default theory and E$\subseteq$F, then E=F.

Normal default theories

Definition Any default of the form

$$\frac{A(\overline{x}) : B(\overline{x})}{B(\overline{x})}$$

is said to be *normal*. A theory $\langle$W,D$\rangle$ is said to be *normal* iff every default from D is normal.

Theorem Every closed normal default theory has an extension.

Theorem If a closed normal default theory has distinct extensions E and F, then E$\cup$F is inconsistent.

Example Consider:

W=$\{\}$ D=$\left\{\frac{:p}{\neg r}, \frac{:r}{\neg p}\right\}$.

T has two extensions:

E=Th($\{\neg r\}$) and F=Th($\{\neg p\}$).

Although E$\cup$F is consistent, $\neg p$ and $\neg r$ should not be regarded as coexisting beliefs.

Theorem (Semi-Monotonicity) Let D and D' be sets of closed normal defaults such that D'$\subseteq$D. Let E' be an extension of T'=$\langle$W,D'$\rangle$ and suppose that T=$\langle$W,D$\rangle$. Then T has an extension E such that E'$\subseteq$E.

What the above theorem says is that any belief derivable from a closed normal theory T cannot be invalidated by augmenting T by a new set of closed normal defaults.

Representational Issues

To what extend normal defaults are sufficient for the purpose of practical applications?

Bill is a high school dropout.

Typically, high school dropouts are adults.

Typically, adults are employed.

W=$\{Dropout(Bill)\}$

D=$\left\{\frac{Dropout(x) : Adult(x)}{Adult(x)}, \frac{Adult(x) : Employed(x)}{Employed(x)}\right\}$.

This theory forces us to believe that Bill is employed.

$$\frac{Adult(x) : Employed(x) \wedge \neg Dropout(x)}{Employed(x)}$$

Bill is a 21-year-old.

All 21-year-olds are adults.

Typically, adults are married.

$21 - Year - Old(Bill)$

$\forall x. 21 - Year - Old(x) \supset Adult(x)$

$$\frac{Adult(x) : Married(x)}{Married(x)}.$$

This theory forces the conclusion that Bill is married.

$$\frac{Adult(x) : Married(x) \wedge \neg 21 - Year - Old(x)}{Married(x)}$$

The general situation: We are given a rule which in isolation is naturally represented by a normal default

$$\frac{A(\overline{x}) : B(\overline{x})}{B(\overline{x})}.$$

When the rule is embedded in the context, there are some exceptional circumstances, say $E(\overline{x})$, which make the rule inapplicable. The simplest solution is to replace the normal default by

$$\frac{A(\overline{x}) : B(\overline{x}) \wedge \neg E(\overline{x})}{B(\overline{x})}.$$

Definition Any default of the form

$$\frac{A(\overline{x}) : B(\overline{x}) \wedge C(\overline{x})}{B(\overline{x})}$$

is called *semi-normal*. A default theory is *semi-normal* iff all of its defaults are semi-normal. Note that a normal default $A : B/B$ can be identified with the semi-normal default $A : B \wedge True/B$.

Semi-normal default theories may lack extensions:

Let $T = \langle \{\}, \mathrm{D} \rangle$, where D consists of

$$\mathrm{d1}=\frac{: p \wedge \neg q}{\neg q}, \qquad \mathrm{d2}=\frac{: q \wedge \neg r}{\neg r}$$

$$\mathrm{d3}=\frac{: r \wedge \neg p}{\neg p}.$$

In general, semi-normal theories lack the property of semi-monotonicity:

Let $T = \langle \{\}, : p \wedge q/q \rangle$. T has one extension:

E=Th($\{q\}$).

Adding $: \neg p/\neg p$ to T, we arrive in a theory with one extension given by

F=Th($\{\neg p\}$).

(1) Are semi-normal theories sufficient to cover all naturally occurring practical applications?

(2) To what extend can the effects obtainable by using semi-normal defaults be achieved in the framework of normal representation?

Translation schema

(1) Replace any default of the form

$$\mathrm{d}=\frac{A(\overline{x}) : B(\overline{x})}{C(\overline{x})}$$

by the semi-normal default

$$\mathrm{d1}=\frac{A(\overline{x}) : B(\overline{x}) \wedge C(\overline{x})}{C(\overline{x})}.$$

This step seems to be intuitively sound what leads to the following observation: **Any common-sense setting which can be correctly modelled by using single-justification defaults may be expressed by means of semi-normal defaults.**

Important: A default and its translation need not be equivalent. Consider d=: $p/\neg p$ and its translation d1=: $p \wedge \neg p/\neg p$. While the theory $\langle\{\}, \{\text{d}\}\rangle$ lacks an extension, the theory $\langle\{\}, \{\text{d1}\}\rangle$ has an extension given by Th({}).

(2) Replace the semi-normal default

$$\text{d1}=\frac{A(\overline{x}) : B(\overline{x}) \wedge C(\overline{x})}{C(\overline{x})}$$

by the normal default

$$\text{d1}=\frac{A(\overline{x}) : B(\overline{x}) \wedge C(\overline{x})}{B(\overline{x}) \wedge C(\overline{x})}.$$

This works if we accept: "Typically, if $A(\overline{x})$ and $C(\overline{x})$, then $B(\overline{x})$.

$$\frac{Adult(x) : Married(x) \wedge \neg 21-Year-Old(x)}{Married(x)}$$

$$\frac{Adult(x) : Married(x) \wedge \neg 21-Year-Old(x)}{Married(x) \wedge \neg 21-Year-Old(x)}$$

Here is an example where the schema does not work.

$$\frac{Has-Motive(x) : Guilty(x)}{Suspect(x)}$$

$$\frac{Has-Motive(x) : Guilty(x) \wedge Suspect(x)}{Suspect(x)}$$

$$\frac{Has-Motive(x) : Guilty(x) \wedge Suspect(x)}{Guilty(x) \wedge Suspect(x)}$$

Advanced topics on default logic, (not disscussed here)

- Semantic characterization of extensions (Łukaszewicz 1985, Etherington 1987).
- Proof theory for DL (Reiter 1980).
- Connections between DL and AEL (Konolige 1988, Marek & Truszczynski 1989).
- Alternative formulations of default logic:
  - Łukaszewicz 1984.
  - Brewka 1991.

Conclusions

Advantages of DL:

Conceptual clarity.

Weaknesses of DL:

(1) Lack of natural semantics.

(2) Some theories lack extensions.

(3) DL cannot deal with reasoning by cases. For instance, given

$$\text{W}=\{bird \vee rocket\}$$

$$\text{D}=\left\{\frac{bird : flies}{flies}, \frac{rocket : flies}{flies}\right\}$$

we intuitively feel that $flies$ should be derivable. Unfortunately, it is not.