Lemma 1 (Ackermann's lemma for propositional logic)
Let p be an atom and α, β be propositional formulae. Let β be positive with respect to p and let α contain no occurrences of p at all.[1] Then the following equivalences hold:

$$\exists p.(p \vee \alpha) \wedge \beta[p \leftarrow \neg p] \equiv \beta[p \leftarrow \alpha] \tag{1}$$

$$\exists p.(\neg p \vee \alpha) \wedge \beta(p) \equiv \beta[p \leftarrow \alpha]. \tag{2}$$

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The algorithm consists of three phases.

Algorithm 1
Phase 1. Preprocessing
The purpose of this phase is to transform the formula $\exists p.\alpha$ into the form that separates positive and negative occurrences of variable p. This form is

$$\exists p.[(\alpha_1(p) \wedge \beta_1(p)) \vee \cdots \vee (\alpha_n(p) \wedge \beta_n(p))] \tag{3}$$

where $\alpha_i(p)$ contains positive and β_i negative ocurrences of the atom p.

1. Eliminate $\Rightarrow$ and $\Leftrightarrow$ and move $\neg$ to the right.
2. Distribute all top-level conjunctions over the disjunctions containing both possitive and negative occurrences of p. For this purpose, apply the equivalences
   - $\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
   - $(\alpha \vee \beta) \wedge \gamma \equiv (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$.

   until the resulting formula is in the form (3).
3. Replace (3) by its equivalent given by

$$\exists p.(\alpha_1(p) \wedge \beta_1(p)) \vee \cdots \vee \exists p.(\alpha_n(p) \wedge \beta_n(p)). \tag{4}$$

[1] A formula β is said to be positive with respect to p iff in conjunctive normal form of β all occurences of p are positive.

Phase 2. Preparation to Ackermann's lemma
The goal of this step is to transform formulae $\exists p.(\alpha_i(p) \wedge \beta_i(p))$ obtained in Phase 1 into one of the forms given by left hand sides of (1) or (2). For this purpose, for each $1 \leq i \leq n$, transform $\alpha_i(p)$ to the form

$$(p \vee \gamma_i) \wedge \delta_i,$$

or $\beta_i(p)$ to the form

$$(\neg p \vee \gamma_i) \wedge \delta_i,$$

where γ_i and δ_i do not contain p. This transformation is always possible using equivalences:

- $(\alpha \wedge \beta) \vee \gamma \equiv (\alpha \vee \gamma) \wedge (\beta \vee \gamma)$
- $\alpha \vee (\beta \wedge \gamma) \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$.

Phase 3. Application of Ackermann's lemma
Repeat this step for each $1 \leq i \leq n$:
Replace

- $\exists p.(p \vee \alpha_i) \wedge \beta_i[p \leftarrow \neg p]$ by $\beta_i[p \leftarrow \alpha_i]$
  or
- $\exists p.(\neg p \vee \alpha_i) \wedge \beta_i(p)$ by $\beta_i[p \leftarrow \alpha_i]$.

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To illustrate the above algorithm, consider the following example. Let's calculate the eliminant

$$\exists p.[((\neg p \Rightarrow q) \wedge \neg p) \vee ((p \wedge q \vee r) \wedge (p \Rightarrow r))].$$

After Phase 1 we obtain

$$\exists p.(p \vee q) \wedge \neg p \vee \exists p.((p \wedge q \vee r) \wedge (\neg p \vee r)).$$

Performing Phase 2 we get

$$\exists p.(p \vee q) \wedge \neg p \vee \exists p.(p \vee r) \wedge (q \vee r) \wedge (\neg p \vee r).$$

Finally, applying Phase 3, we receive $q \vee r$.