10

Formalization of Default Logic Using CAKE

10.1 Introduction

In this chapter, we formalize a subset of default logic using the CAKE method.[1] The goal of this chapter is to do a case study showing how the CAKE method can be used to model a particular type of reasoning commonly used in knowledge representation and important in many applications. This will be done by representing two basic versions of default logic: *rough default logic* and *rough default logic with strong prerequisites*. The main difference between the two versions that will be modeled lies in different treatment of the prerequisite of a default while determining the default's applicability. In the former, a default can be applied if its prerequisite is believed (not contradicting known information). In the latter, we may require that the prerequisite of a default (or a part of it) has to be known, rather than believed, to make the default applicable. The possibility of using both versions substantially increases the expressive power of the resulting logic. We also show that both rough default logic and rough default logic with strong prerequisites can be naturally extended to their prioritized versions by slightly changing the voting policy mechanism used.

It is important to note that the underlying semantics for the considered versions of default logic differ from traditional approaches such as Reiter's default logic.

The following discussion assumes familiarity with the material contained in Chapters 4 and 9.

A default is said to be *disjunction-free* if its prerequisite and justification are conjunctions of literals and its consequent is a single literal. A default theory is said to be *disjunction-free* if all its defaults are disjunction-free and all its axioms are ground literals.

[1] Recall that default logic has been presented in Section 4.2, Chapter 4.

In the sequel, we restrict ourselves to finite disjunction-free default theories with consistent sets of axioms.[2] We assume that the axioms of a theory are ground literals extracted from an underlying extensional database(s).[3]

We will start by modeling and discussing a number of standard default theories found in the literature and provide a means of comparing our underlying semantics with the more traditional approaches found in the literature. We will then provide the technical framework and formal semantics based on the use of CAKE.

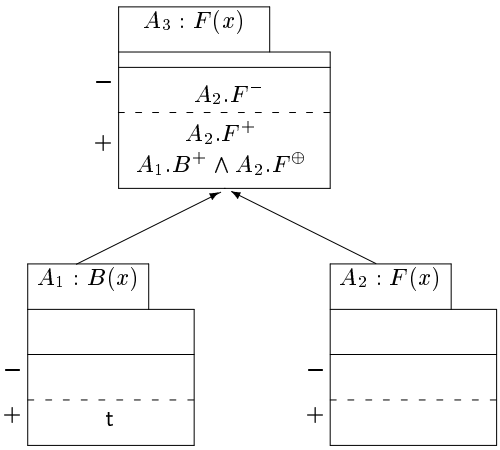

Fig. 10.1. Diagram corresponding to the theory of Example 10.1.1.

Example 10.1.1. Consider the theory $T = \langle \{B(\mathsf{t})\}, \{B(x) : F(x)/F(x)\} \rangle$. This is the standard "Bird" theory with B, F, t denoting *Bird*, *Flies* and Tweety, respectively. Figure 10.1 shows a CAKE diagram corresponding to the theory T.[4]

The bottom granules, A_1 and A_2, are CAKE granules representing data from an assumed extensional database. These granules are responsible for the relations B and F, respectively. Note that t is in the positive part of the diagram of A_1 and in the boundary region of the diagram of A_2. The granule A_3 represents the default rule of the theory. It is responsible for the relation F

[2] Note that the task of determining whether a set of axioms of a given disjunction-free default theory is consistent is trivial.

[3] That is, ground literals of the form $R(\overline{a})$ (respectively $\neg R(\overline{a})$), where $\overline{a}$ is in the positive (respectively negative) part of the diagram of the extensional database granule responsible for the relation R.

[4] Recall that for the sake of simplicity we often omit variables of relations in CAKE diagrams, when these are known from the context.

which occurs in the default consequent. There are three methods associated with this granule. The first two, namely $A_2.F^-$ and $A_2.F^+$, allows the granule A_3 to use knowledge of the granule A_2 while computing the relation F. Using these methods the CAKE granule A_3 can infer that, for any object x, x lies in the negative (respectively positive) part of his diagram, provided that x lies in the negative (respectively positive) part of the diagram of A_2. The third method, i.e., $A_1.B^+ \wedge A_2.F^{\oplus}$, represents the default of the theory. Note that the method is placed in the positive part of the diagram. It is to be viewed as the following rule:

> "For any object x, if x is in the positive part of the diagram of A_1 and x is in the positive/boundary part of the diagram of A_2, infer that x is in the positive part of the diagram of A_3."

The following set of rules is associated with the above diagram:[5]

$$A_2.F^-(x) \rightarrow A_3.F^-(x)$$
$$\{A_2.F^+(x) \vee [A_1.B^+(x) \wedge A_2.F^{\oplus}(x)]\} \rightarrow A_3.F^+(x)$$
$$A_1.B^+(\mathsf{t}).$$

Eliminating the reference to the boundary region in $A_2.F^{\oplus}(x)$, i.e., replacing $A_2.F^{\oplus}(x)$ by $\neg A_2.F^-(x)$ (see Chapter 9, Section 9.4), results in the following modified rule set:

$$A_2.F^-(x) \rightarrow A_3.F^-(x) \tag{10.1}$$
$$\{A_2.F^+(x) \vee [A_1.B^+(x) \wedge \neg A_2.F^-(x)]\} \rightarrow A_3.F^+(x) \tag{10.2}$$
$$A_1.B^+(\mathsf{t}). \tag{10.3}$$

Clearly, the partition $\{(10.3)\}, \{(10.1), (10.2)\}$ provides a stratification of the above set of rules.

The relations $A_3.F^-$, $A_3.F^+$ and $A_1.B^+$ occurring in the heads of the rules are specified by a simultaneous fixpoint expression of the form

$$\text{LFP}\, A_3.F^-, A_3.F^+, A_1.B^+.\mathcal{B}, \tag{10.4}$$

where $\mathcal{B}$ is the conjunction of the rules' bodies.[6]

Below, we compute the relations characterized by (10.4), using a slight generalization of Algorithm 3.3.2 from Chapter 3. The successive lines consist of sets of literals obtained after performing all successive iterations.

$$\{\}$$
$$\{A_1.B^+(\mathsf{t})\}$$
$$\{A_1.B^+(\mathsf{t}), A_3.F^+(\mathsf{t})\}.$$

Suppose the query we are interested in is $F(\mathsf{t})$. It is immediately seen that t satisfies the $A_3.F^+$-coordinate of (10.4). Consequently, the answer to the query is TRUE. ∎

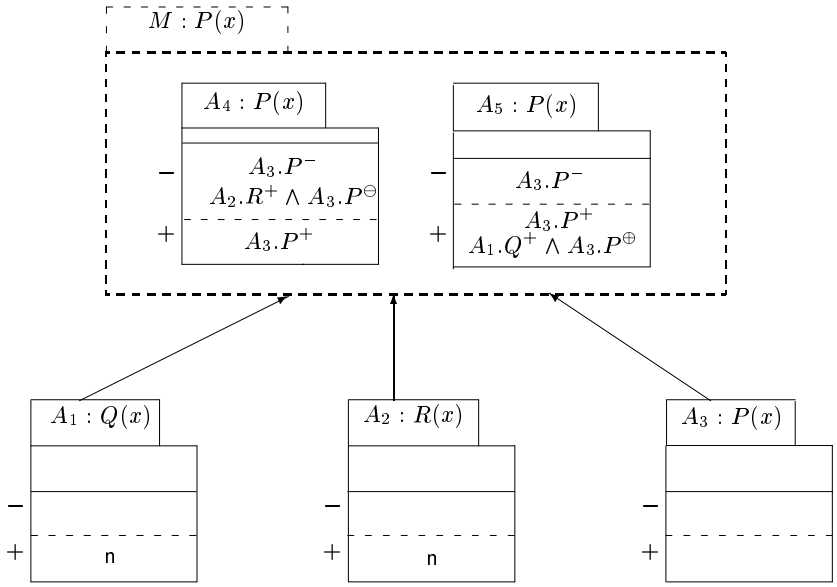

Fig. 10.2. Diagram corresponding to the theory of Example 10.1.2.

Example 10.1.2. Consider the theory

$$T = \left\langle \{Q(\mathsf{n}), R(\mathsf{n})\}, \left\{ \frac{R(x) : \neg P(x)}{\neg P(x)}, \frac{Q(x) : P(x)}{P(x)} \right\} \right\rangle .$$

This is the standard "Nixon" diamond theory with R, P, Q, n standing for *Republican*, *Pacifist*, *Quaker* and Nixon, respectively (see Example 4.2.5). Figure 10.2 shows a CAKE diagram corresponding to the theory T.

The granules A_1, A_2 and A_3 represent data from an assumed extensional database, whereas A_4 and A_5 represent the defaults of the theory.[7] Since both A_4 and A_5 make inferences concerning the relation P, they have been grouped

[5] Recall that all free variables occurring in the rules are implicitly universally quantified.

[6] See Chapter 9, Section 9.5.4.

[7] Observe that the method $A_2.R^+ \wedge A_3.P^{\ominus}$ used by the granule A_4 is placed in the negative part of its diagram. Accordingly, it has the following reading: "for any object x, if x is in the positive part of the diagram of A_2 and x is in the negative/boundary part of the diagram of A_3, infer that x is in the negative part of the diagram of A_4."

into a single default module.[8] It is the module, not an individual granule, that is responsible for default inferences about P.

The following set of rules is associated with the above diagram:[9]

$A_1.Q^+(\mathsf{n})$
$A_2.R^+(\mathsf{n})$
$\{A_3.P^-(x) \vee [A_2.R^+(x) \wedge \neg A_3.P^+(x)]\} \rightarrow A_4.P^-(x)$
$A_3.P^+(x) \rightarrow A_4.P^+(x)$
$A_3.P^- \rightarrow A_5.P^-(x)$
$\{A_3.P^+(x) \vee [A_1.Q^+(x) \wedge \neg A_3.P^-(x)]\} \rightarrow A_5.P^+(x)$
$[A_4.P^-(x) \vee A_5.P^-(x)] \wedge \neg[A_4.P^+(x) \vee A_5.P^+(x)] \rightarrow M.P^-(x)$
$[A_4.P^+(x) \vee A_5.P^+(x)] \wedge \neg[A_4.P^-(x) \vee A_5.P^-(x)] \rightarrow M.P^+(x).$

The last two rules represent the standard voting mechanism used by the module M.

The above set of rules is clearly stratified by the partition P_1, P_2, P_3, where P_1 consists of the first two rules, P_3 consists of the last two rules and P_2 contains the remaining rules. The relations $A_1.Q^+$, $A_2.R^+$, $A_4.P^-$, $A_4.P^+$, $A_5.P^-$, $A_5.P^+$, $M.P^-$ and $M.P^+$, occurring in the heads of the above rules, are defined by the fixpoint expression given by

$$\text{LFP}\, A_1.Q^+, A_2.R^+, A_4.P^-, A_4.P^+, A_5.P^-, A_5.P^+, M.P^-, M.P^+.\mathcal{B} \qquad (10.5)$$

where $\mathcal{B}$ denotes the conjunction of the rules' bodies.

Below, we compute the relations characterized by the expression (10.5).

$\{\}$
$\{A_1.Q^+(\mathsf{n}), A_2.R^+(\mathsf{n})\}$
$\{A_1.Q^+(\mathsf{n}), A_2.R^+(\mathsf{n}), A_4.P^-(\mathsf{n}), A_5.P^+(\mathsf{n})\}.$

Suppose the query of interest is $P(\mathsf{n})$. It is immediately seen that the granule A_4 answers FALSE and the granule A_5 answers TRUE to the query. However, since it is the default module that is responsible for default inferences about P, and since n satisfies neither the $M.P^+$-coordinate of (10.5) nor the $M.P^-$-coordinate of (10.5), we conclude that the answer to the query $P(\mathsf{n})$ is UNKNOWN. ∎

The examples considered so far are typified by the fact that the CAKE granules that represent default rules obtain all relevant input data from an assumed extensional database. The next example shows that the situation can be more complex.

[8] In Chapter 3, a collection of CAKE granules was referred to as a knowledge module. Here we use the term "a default module," since all granules included in such a module represent defaults.

[9] The relations $A_3.P^{\ominus}$ and $A_3.P^{\oplus}$, occurring in the diagram, have been replaced by $\neg A_3.P^+$ and $\neg A_3.P^-$, respectively.

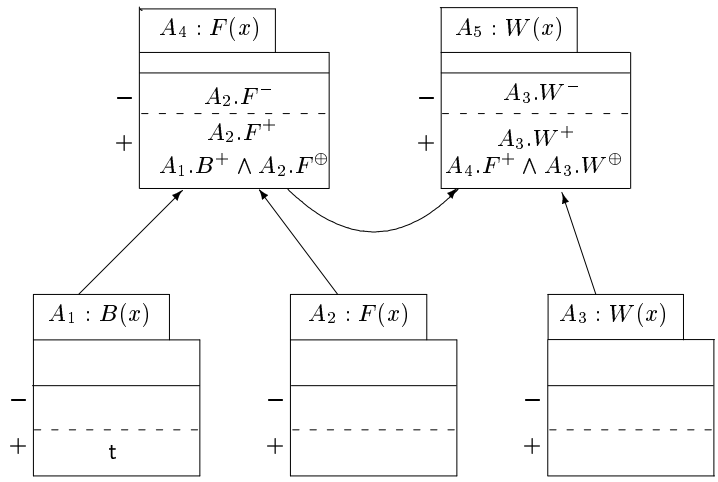

Fig. 10.3. Diagram corresponding to the theory of Example 10.1.3.

Example 10.1.3. Consider the theory T given by

$$T = \left\langle \{B(\mathsf{t})\}, \left\{ \frac{B(x) : F(x)}{F(x)}, \frac{F(x) : W(x)}{W(x)} \right\} \right\rangle,$$

where B, F, W and t denote $Bird$, $Flies$, $HasWings$ and Tweety, respectively.

The corresponding CAKE diagram for T is provided in Figure 10.3.

Note that there is an arrow between A_4 and A_5. This is because A_4 is responsible for the relation F that is used by A_5 while computing the relation W. In this case, not all the base data input to the default granules comes from an extensional database as in the previous examples.

The following set of rules are associated with the diagram.

$A_1.B^+(\mathsf{t})$
$A_2.F^-(x) \rightarrow A_4.F^-(x)$
$\{A_2.F^+(x) \vee [A_1.B^+(x) \wedge \neg A_2.F^-(x)]\} \rightarrow A_4.F^+(x)$
$A_3.W^-(x) \rightarrow A_5.W^-(x)$
$\{A_3.W^+(x) \vee [A_4.F^+(x) \wedge \neg A_3.W^-(x)]\} \rightarrow A_5.W^+(x).$

The partition P_1, P_2, where P_1 consists of the first rule and P_2 contains the remaining rules, provides a stratification of the above set.

The relations $A_1.B^+$, $A_4.F^-$, $A_4.F^+$, $A_5.W^-$ and $A_5.W^+$, occurring in the heads of the rules, are defined by the fixpoint expression given by

$$\text{LFP}\, A_1.B^+, A_4.F^-, A_4.F^+, A_5.W^- A_5.W^+.\mathcal{B} \tag{10.6}$$

where $\mathcal{B}$ denotes the conjunction of the rules' bodies.

The computation of the relations characterized by (10.6) is given below.

$\{\}$

$\{A_1.B^+(\mathsf{t})\}$

$\{A_1.B^+(\mathsf{t}), A_4.F^+(t)\}$

$\{A_1.B^+(\mathsf{t}), A_4.F^+(\mathsf{t}), A_5.W^+(\mathsf{t})\}$.

Suppose the query we are interested in is $W(\mathsf{t})$. Since t satisfies the $A_5.W^+$-coordinate of fixpoint (10.6), we conclude that the answer to the query $W(\mathsf{t})$ is TRUE. ∎

The next example shows that this version of default logic does not admit ungrounded conclusions.

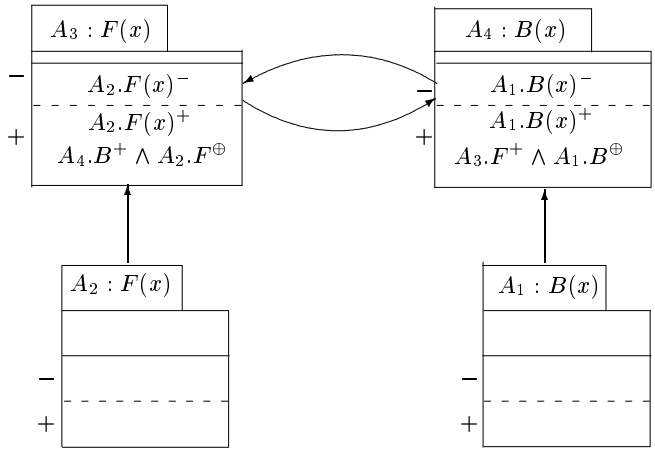

Fig. 10.4. Diagram corresponding to the theory of Example 10.1.4.

Example 10.1.4. Consider the theory

$$T = \left\langle \emptyset, \left\{ \frac{B(x) : F(x)}{F(x)}, \frac{F(x) : B(x)}{B(x)} \right\} \right\rangle,$$

where B and F stand for *Bird* and *Flies*, respectively. A CAKE diagram for the theory T is provided in Figure 10.4.

The following set of rules is associated with the diagram:

$A_2.F^-(x) \rightarrow A_3.F^-(x)$

$\{A_2.F^+(x) \vee [A_4.B^+(x) \wedge \neg A_2.F^-(x)]\} \rightarrow A_3.F^+(x)$

$A_1.B^-(x) \rightarrow A_4.B^-(x)$

$\{A_1.B^+(x) \vee [A_3.F^+(x) \wedge \neg A_1.B^-(x)]\} \rightarrow A_4.B^+(x)$.

The set of rules shown above is stratified using a single partition P_1, where P_1 consists of all the rules.

The relations occurring in the heads of the rules are specified by the fixpoint expression of the form

$$\text{LFP}\, A_3.F^-, A_3F^+, A_4.B^-, A_4.B^+.\mathcal{B}, \tag{10.7}$$

where $\mathcal{B}$ denotes the conjunction of the rules' bodies.

Suppose that the query we are interested in is $B(\mathsf{t})$. It is immediately seen that all the relations characterized by the expression (10.7) have empty extensions.[10] Accordingly, the answer to the query is UNKNOWN. ∎

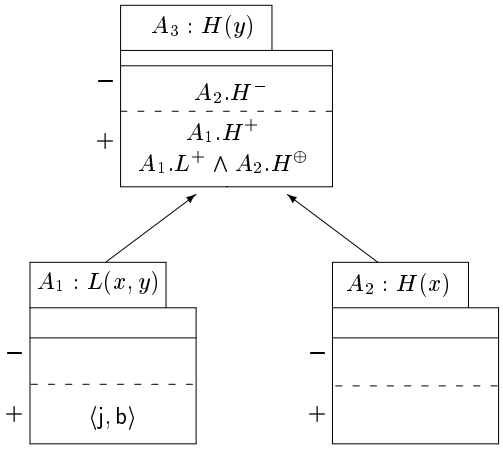

Fig. 10.5. Diagram corresponding to the theory of Example 10.1.5.

Example 10.1.5. Consider the theory

$$T = \left\langle \{L(\mathsf{j},\mathsf{b})\}, \left\{ \frac{L(x,y) : H(y)}{H(y)} \right\} \right\rangle,$$

where L, H, j and b denote *Likes*, *Happy*, John and Bill, respectively. The CAKE diagram corresponding to the theory T is shown in Figure 10.5.

The rules associated with the diagram are the following.

$A_1.L^+(\mathsf{j},\mathsf{b})$

$A_2.H^-(x) \rightarrow A_3.H^-(x)$

$\exists y.\, \{A_2.H^+(x) \vee [A_1.L^+(x,y) \wedge \neg A_2.H^-(x)]\} \rightarrow A_3.H^+(x).$

[10] Recall that the extension of an n-ary relation is the set of all n-tuples satisfying this relation.

The partition P_1, P_2, where P_1 consists of the first rule and P_2 contains the remaining rules provides a stratification of the above set.

The relations occurring in the heads of the rules are defined by the fixpoint expression given by

$$\text{LFP}\, A_1.L^+, A_3.H^-, A_3.H^+.\mathcal{B}, \tag{10.8}$$

where, as usual, $\mathcal{B}$ is the conjunction of all the rules' bodies.

The computation for the relations mentioned in formula (10.8) are provided below.

$\{\}$

$\{A_1.L^+(\mathsf{j},\mathsf{b})\}$

$\{A_1.L^+(\mathsf{j},\mathsf{b}), A_3.H^+(\mathsf{b})\}$.

Suppose that the query under consideration is $H(\mathsf{b})$. Since b satisfies the $A_3.H^+$-coordinate of (10.8), the answer to the query $H(\mathsf{b})$ is TRUE. ∎

10.2 Computing Normal Default Theories

We refer to the version of default logic exemplified in the previous section as *rough default logic*. In this section, we begin the presentation of the technical details by first considering normal default theories.[11]

We begin with some terminology. Two defaults are said to be *similar* if their consequents contain the same relation symbol. The following rather detailed definition describes the construction of CAKE diagrams for arbitrary disjunction-free normal default theories.

Definition 10.2.1. *Let $T = \langle W, D\rangle$ be a normal default theory over a language $\mathcal{L}$. A* normal default CAKE diagram *corresponding to a normal default theory T is constructed as follows.*

1. *For each relation symbol R occurring in $\mathcal{L}$, we construct a* CAKE *granule, called a* database granule*, representing R. If $R(\overline{t})$ (respectively $\neg R(\overline{t})$) is in W, the tuple $\overline{t}$ occurs in the positive (respectively negative) part of the diagram of the granule. All other tuples of constants are assumed to be in the boundary region of the diagram.*[12] *Each database granule is labeled by name: $R(-)$, where name is a unique name of the granule and R is the relation symbol it is responsible for.*
2. *For each default d from D, we construct a* CAKE *granule, called a* default granule*, representing d. The granule is labeled by name: R, where name is a unique name of the granule and R is the relation symbol occurring in the consequent of d. Methods used by default granules will be specified later.*

[11] Recall that all the theories considered in this chapter are disjunction-free default theories with consistent sets of axioms.

[12] Recall that the boundary part of a diagram is not explicitly drawn.

3. *All default granules, say $A_1, \ldots, A_k$, where $k > 1$, representing similar defaults are grouped into a single knowledge module, referred to as a* default module. *The module is labeled by name: R, where name is a unique name of the module and R is the relation symbol the granules $A_1, \ldots, A_k$ are responsible for. In what follows, any default granule which is not embedded in a default module will be referred to as an* independent default granule. *Otherwise, the granule will be referred to as a* dependent default granule.
4. *Let A be an independent default granule representing a default d and suppose that the consequent of d contains a relation symbol R. Assume further that $R_1, \ldots, R_k$ are all the relation symbols occurring in the prerequisite of d. For each $1 \leq i \leq k$, we create an arrow from a database granule representing R_i to A and an arrow from a module (or independent default granule if it exists) representing R_i. We also create an arrow from a database granule representing R to A.*
5. *Let M be a module representing a relation R. Let $A_1, \ldots, A_n$ be default granules of M and suppose that A_i corresponds to a default d_i. Assume further that $R_1, \ldots, R_k$ are all the relation symbols occurring in the prerequisites of $d_1, \ldots, d_n$. For each $1 \leq i \leq k$, we create an arrow from a database granule representing R_i to M and, an arrow from a module (or independent default granule if it exists) representing R_i. We also create an arrow from a database granule representing R to M.*
6. *For each default granule (dependent or not) we now assign the methods it will employ. Assume that the granule represents a default of the form $l_1 \wedge \cdots \wedge l_n : l/l$, where $n \geq 0$, and suppose that $R_1, \ldots, R_n, R$ are the relation symbols occurring in $l_1, \ldots, l_n, l$, respectively. First of all, there are two methods, namely $A_k.R^+$ and $A_k.R^-$, where A_k is the database granule responsible for the relation R. These methods, which will be referred to as database methods, are placed in the positive and negative parts of the diagram, respectively. In addition, there is one method corresponding to the default represented by the granule. This method, referred to as a* default method, *is placed in the positive part of the granule's diagram if l is a positive literal and in the negative part, otherwise. The default method is constructed as follows.*
   a) *Suppose first that none of $R_1, \ldots, R_n$ is represented either by a default module or by an independent default granule. Then the default method is given by*
   $$A_1.X_1 \wedge \ldots \wedge A_n.X_n \wedge A.X, \tag{10.9}$$
   where $A_1, \ldots, A_n, A$ are database granules responsible for $R_1, \ldots, R_n, R$, respectively, each X_i is R_i^+ (if l_i is positive) or R_i^- (if l_i is negative) and X is $R^\oplus$ (if l is positive) or $R^\ominus$ (if l is negative).
   b) *Suppose now that at least one of $R_1, \ldots, R_n$ is represented by a default module or by an independent default granule. Let $R_i, \ldots R_j$ be all relations from $R_1, \ldots, R_n$ that are represented by default modules (or independent default granules) and suppose that $M_i, \ldots, M_j$ are*

the corresponding modules (or independent default granules). Then the default method is (10.9) with $A_i, \ldots, A_j$ replaced by $M_i, \ldots, M_j$, respectively. ∎

The set of rules corresponding to a diagram of a default theory is specified using CAKE's standard methodology see Chapter 9, Section 9.5.3). We also use the standard voting policy to resolve conflicts, i.e., adjudicating granules are not included in default modules.

The following example shows that the rules attached to diagrams of normal default theories are not always stratified.

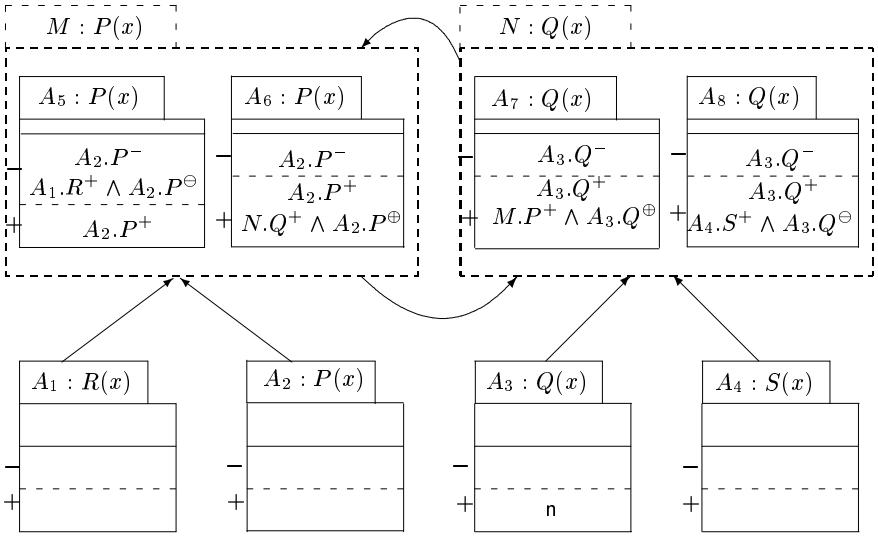

Fig. 10.6. Diagram corresponding to the theory of Example 10.2.2.

Example 10.2.2. Consider the theory $T = \langle W, D \rangle$, where $W = \{Q(\mathsf{n})\}$ and

$$D = \left\{ \frac{Q(x) : P(x)}{P(x)}, \frac{R(x) : \neg P(x)}{\neg P(x)}, \frac{P(x) : Q(x)}{Q(x)}, \frac{S(x) : \neg Q(x)}{\neg Q(x)} \right\}.$$

The diagram of this theory is shown in Figure 10.6. The set of rules attached to the diagram is:

$$A_3.Q^+(\mathsf{n}) \tag{10.10}$$

$$[A_2.P^-(x) \vee (A_1.R^+(x) \wedge \neg A_2.P^+(x))] \rightarrow A_5.P^-(x) \tag{10.11}$$

$$A_2.P^+(x) \rightarrow A_5.P^+(x) \tag{10.12}$$

$$A_2.P^-(x) \rightarrow A_6.P^-(x) \tag{10.13}$$

$$[A_2.P^+(x) \vee (N.Q^+(x) \wedge \neg A_2.P^-(x))] \rightarrow A_6.P^+(x) \quad (10.14)$$
$$A_3.Q^-(x) \rightarrow A_7.Q^-(x) \quad (10.15)$$
$$[A_3.Q^+(x) \vee (M.P^+(x) \wedge \neg A_3.Q^-(x))] \rightarrow A_7.Q^+(x) \quad (10.16)$$
$$A_3.Q^-(x) \rightarrow A_8.Q^-(x) \quad (10.17)$$
$$[A_3.Q^+(x) \vee (A_4.S^+(x) \wedge \neg A_3.Q^-(x))] \rightarrow A_8.Q^+(x) \quad (10.18)$$
$$[A_5.P^-(x) \vee A_6.P^-(x)] \wedge \neg[A_5.P^+(x) \vee A_6.P^+(x)] \rightarrow M.P^-(x) \quad (10.19)$$
$$[A_5.P^+(x) \vee A_6.P^+(x)] \wedge \neg[A_5.P^-(x) \vee A_6.P^-(x)] \rightarrow M.P^+(x) \quad (10.20)$$
$$[A_7.Q^-(x) \vee A_8.Q^-(x)] \wedge \neg[A_7.Q^+(x) \vee A_8.Q^+(x)] \rightarrow N.Q^-(x) \quad (10.21)$$
$$[A_7.Q^+(x) \vee A_8.Q^+(x)] \wedge \neg[A_7.Q^-(x) \vee A_8.Q^-(x)] \rightarrow N.Q^+(x). \quad (10.22)$$

It can easily be observed that the above rules are not stratified. ∎

In what follows, we mainly deal with stratified default theories as defined below. However, non-stratified default theories will be briefly discussed in Section 10.6.

Definition 10.2.3. *By a* stratified default theory *(normal or not) we understand a theory whose* CAKE *diagram leads to a stratified set of rules.* ∎

The following definition of provability will be used shortly.

Definition 10.2.4. *Let T be a stratified normal default theory and suppose that* **R** *is the set of rules corresponding to the diagram of T. Let $\mathcal{B}$ be the conjunction of the rules' bodies from* **R** *and assume that $R_1, \ldots, R_k$ are all relation symbols occurring in the heads of the rules. Consider the fixpoint expression given by*

$$\text{LFP}\, R_1, \ldots, R_k.\mathcal{B}. \quad (10.23)$$

A ground literal of the form $P(\overline{c})$ (respectively $\neg P(\overline{c})$) is said to be provable *in T if and only if $\overline{c}$ satisfies the R_i-coordinate of (10.23), R_i is of the form $A.P^+$ (respectively $A.P^-$) and A is the name associated with a database granule, an independent default granule or a default module.* ∎

The central concept of default logic is that of an extension of a default theory. In rough default logic this notion can be defined as follows.

Definition 10.2.5. *Let T be a stratified normal default theory. An* extension *of T is the set of all ground literals which are provable in T.* ∎

In view of earlier examples, one may suspect that a ground literal $l(\overline{t})$ belongs to an extension of a stratified normal default theory T if and only if $l(\overline{t})$ is a member of every extension of T in Reiter's default logic. However, as the next example shows, this is not always the case.

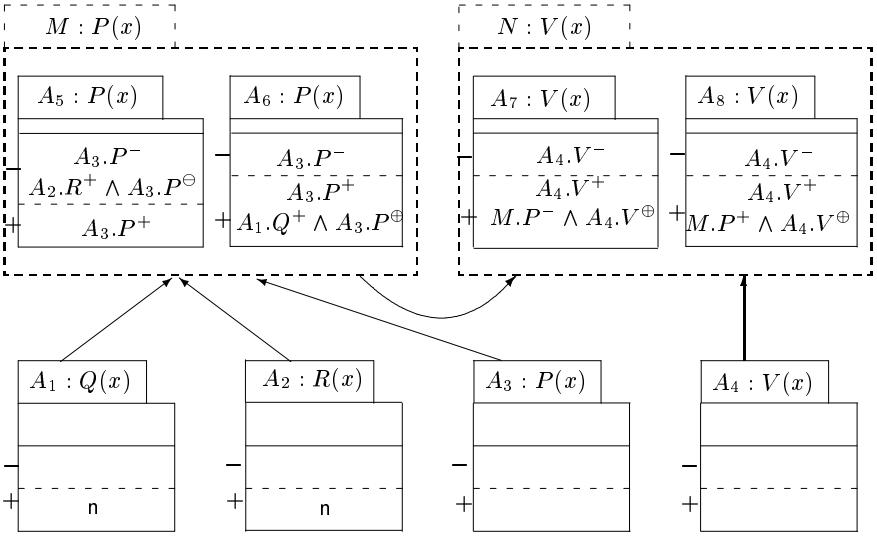

Fig. 10.7. Diagram corresponding to the theory of Example 10.2.6.

Example 10.2.6. Consider the theory $T = \langle W, D \rangle$, where

$$W = \{Q(\mathsf{n}), R(\mathsf{n})\}$$

$$D = \left\{ \frac{Q(x) : P(x)}{P(x)}, \frac{R(x) : \neg P(x)}{\neg P(x)}, \frac{P(x) : V(x)}{V(x)}, \frac{\neg P(x) : V(x)}{V(x)} \right\}.$$

This is an extended version of the standard "Nixon" diamond theory where R, P, Q, V, n denote *Republican*, *Pacifist*, *Quaker*, *Voter* and Nixon, respectively. The CAKE diagram corresponding to the theory T is shown in Figure 10.7.

In Reiter's default logic T has two extensions, E_1 and E_2, given by

$$E_1 = Cn(\{Q(\mathsf{n}), R(\mathsf{n}), V(\mathsf{n}), P(\mathsf{n})\})$$
$$E_2 = Cn(\{Q(\mathsf{n}), R(\mathsf{n}), V(\mathsf{n}), \neg P(\mathsf{n})\}).$$

Note that the literal $V(\mathsf{n})$ is a member of both E_1 and E_2. However, in rough default logic the literal $V(\mathsf{n})$ does not belong to the extension. To see this, consider the set of rules associated with the diagram of the theory T:

$$A_1.Q^+(\mathsf{n})$$
$$A_2.R^+(\mathsf{n})$$
$$\{A_3.P^-(x) \vee [A_2.R^+(x) \wedge \neg A_3.P^+(x)]\} \rightarrow A_5.P^-(x)$$
$$A_3.P^+(x) \rightarrow A_5.P^+(x)$$
$$A_3.P^-(x) \rightarrow A_6.P^-(x)$$

$$\{A_3.P^+(x) \vee [A_1.Q^+(x) \wedge \neg A_3.P^-(x)]\} \rightarrow A_6.P^+(x)$$
$$A_4.V^-(x) \rightarrow A_7.V^-(x)$$
$$\{A_4.V^+(x) \vee [M.P^-(x) \wedge \neg A_4.V^-(x)]\} \rightarrow A_7.V^+(x)$$
$$A_4.V^-(x) \rightarrow A_8.V^-(x)$$
$$\{A_4.V^+(x) \vee [M.P^+(x) \wedge \neg A_4.V^-(x)]\} \rightarrow A_8.V^+(x)$$
$$[A_5.P^-(x) \vee A_6.P^-(x)] \wedge \neg[A_5.P^+(x) \vee A_6.P^+(x)] \rightarrow M.P^-(x)$$
$$[A_5.P^+(x) \vee A_6.P^+(x)] \wedge \neg[A_5.P^-(x) \vee A_6.P^-(x)] \rightarrow M.P^+(x)$$
$$[A_7.V^-(x) \vee A_8.V^-(x)] \wedge \neg[A_7.V^+(x) \vee A_8.V^+(x)] \rightarrow N.V^-(x)$$
$$[A_7.V^+(x) \vee A_8.V^+(x)] \wedge \neg[A_7.V^-(x) \vee A_8.V^-(x)] \rightarrow N.V^+(x).$$

The last four rules represent the standard voting mechanism used by the modules M and N.

The above set of rules is clearly stratified.

The relations occurring in the heads of the above rules are defined by the following fixpoint expression:

$$\mathrm{LFP}\,\overline{X}.\mathcal{B}, \tag{10.24}$$

where $\overline{X}$ is the tuple $A_1.Q^+, A_2R^+, A_5.P^-, A_5.P^+, A_6.P^-, A_6.P^+, A_7.V^-$, $A_7.V^+$, $A_8.V^-, A_8.V^+, M.P^-, M.P^+, N.V^-, N.V^+$ and $\mathcal{B}$ is the conjunction of all the rules' bodies.

The computation of the relations defined by the formula (10.24) is given below.

$$\{\}$$
$$\{A_1.Q^+(\mathsf{n}), A_2.R^+(\mathsf{n})\}$$
$$\{A_1.Q^+(\mathsf{n}), A_2.R^+(\mathsf{n}), A_5.P^-(\mathsf{n}), A_6.P^+(\mathsf{n})\}.$$

Now, it is immediately seen that n does not satisfy the $N.V^+$-coordinate of (10.24). Accordingly, $V(\mathsf{n})$ is not provable in the theory T and hence $V(\mathsf{n})$ does not belong to the extension of T. ∎

The above examples should provide some intuitions as to how one models and uses normal default theories in CAKE. The crucial observation is that defaults whose consequents refer to the same relation symbol are grouped in a module. In contrast to Reiter's logic, where these defaults can interact individually, in this formalism the interaction is achieved via modules.

The next two results follow straightforwardly.

Theorem 10.2.7. *In rough default logic every stratified normal default theory has one extension.* ∎

Theorem 10.2.8. *In rough default logic every stratified normal default theory with consistent set of axioms has a consistent extension.* ∎

Observe that the property of semi-monotonicity generally does not hold in rough default logic. To see this, consider the normal theory

$$T = \left\langle \{Q(\mathsf{n}), R(\mathsf{n})\}, \left\{ \frac{Q(x) : P(x)}{P(x)} \right\} \right\rangle.$$

The extension of this theory is $E = \{Q(\mathsf{n}), R(\mathsf{n}), P(\mathsf{n})\}$.

Now, if we add a new default $\dfrac{R(x) : \neg P(x)}{\neg P(x)}$, the extension of the expanded theory is $E' = \{Q(\mathsf{n}), R(\mathsf{n})\}$.

10.3 Default Logic with Strong Prerequisites

Both standard default logic and the approach introduced in section 10.2 lack tools to distinguish between facts *known* to be true, i.e., derivable from axioms, and those *believed* to be true, i.e., derivable by applying defaults. In consequence, both these formalisms are unable to properly deal with many practically occurring settings. The next example will help to illustrate this.

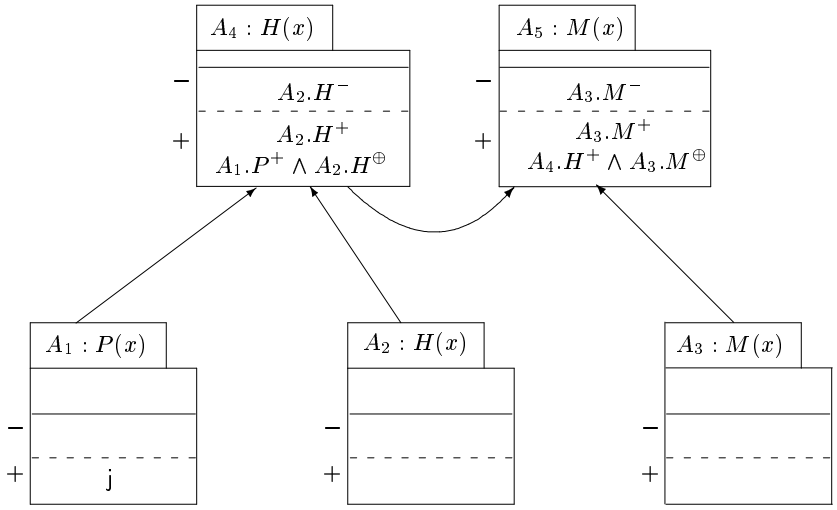

Fig. 10.8. Diagram corresponding to the theory of Example 10.3.1.

Example 10.3.1. Suppose we are given the following facts:

John is a person.
Normally, a person is honest.
Normally, a person known to be honest can be safely lent money.

Given these facts, we are prepared to believe that John is honest, but we do not want to infer that he can be safely lent money. The reason, of course, is that John's honesty is a default conclusion rather, than an iron-clad fact.

This commonsense theory cannot be properly formalized according to the intuitions stated above in the framework of default logic introduced in section 10.2. The best one can do is to use the following default theory T given by

$$W = \{P(\mathsf{j})\} \qquad D = \left\{ \frac{P(x) : H(x)}{H(x)}, \ \frac{H(x) : M(x)}{M(x)} \right\},$$

where P, H and M and j stand for "Person," "Honest," "Can be safely lent money" and "John," respectively.

The CAKE diagram for theory T is provided in Figure 10.8.

It is easily checked that, contrary to our intuition, the literal $M(\mathsf{j})$ is a member of the extension of T.

Fortunately, the CAKE method provides us with a simple way to block the unwanted conclusion which also makes intuitive sense. All that has to be done is to remove the arrow from A_4 to A_5, create an additional arrow from A_2 to A_5 and replace the method $A_4.H^+ \wedge A_3.M^{\oplus}$, assigned to A_5, by $A_2.H^+ \wedge A_3.M^{\oplus}$ (see Figure 10.9).

In this representation, the second default is inapplicable for John because his honesty cannot be inferred directly from the database. ∎

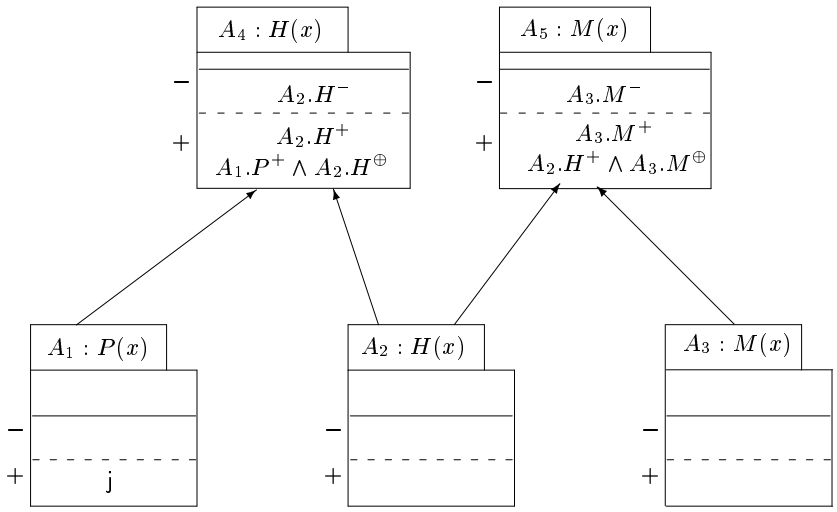

Fig. 10.9. Modified diagram corresponding to the theory of Example 10.3.1.

We now formalize the above idea in a systematic manner by introducing a new variant of rough default logic. This variant, called *rough default logic with strong prerequisites* (SP *rough default logic*, for short), explicitly distinguishes between facts known to be true and those believed to be true. Facts known to be true are simply those atomic facts that are represented explicitly in an assumed extensional database.

We extend the language of rough default logic by adding a new operator **K** which can be applied to literals. A formula of the form $\mathbf{K}l$ is to be read as "l is known to be true."[13]

As stated previously, we limit ourselves to disjunction-free default theories. However, in SP rough default logic any conjunct occurring in the prerequisite a of a default can be preceded by the operator **K**. More specifically, we consider defaults of the form

$$\frac{B_1 \wedge \cdots \wedge B_n : l}{l}$$

where each B_i is either a literal or a literal preceded by the operator **K** and l is a literal.

The formal difference between rough and SP rough default logic lies in the definition of a diagram. In the latter, any conjunct of the form $\mathbf{K}l$ occurring in the prerequisite of a default must be supported by the axioms of the theory under consideration, rather than by the consequent of an applied default.

Definition 10.3.2. *Let T be a disjunction-free normal default theory of* SP *rough default logic. A diagram corresponding to T is constructed as before (see Definition 10.2.1) with points 4, 5 and 6 replaced by:*

4. *Let A be an independent default granule representing a default d and suppose that the consequent of d contains a relation symbol R. Assume further that $R_1,\ldots,$ R_k are all the relation symbols occurring in the prerequisite of d and let $R'_1,\ldots R'_l$ be those elements from $R_1,\ldots,$ R_k that are not preceded by the operator* **K**. *For each $1 \leq i \leq k$, we place an arrow from a database granule representing R_i to A and, for each $1 \leq i \leq l$, we place an arrow from a module (or independent default granule if it exists), representing R'_i to A. We also place an arrow from a database granule representing R to A.*
5. *Let M be a module representing a relation R. Let A_1, ..., A_n be default granules of M and suppose that A_i corresponds to a default d_i. Assume further that $R_1,\ldots,$ R_k are all the relation symbols occurring in the prerequisites of $d_1,\ldots,d_n$ and $R'_1,\ldots,R'_l$ are those elements from $R_1,\ldots,$ R_k that are not preceded by the operator* **K**. *For each $1 \leq i \leq k$, we place an arrow from a database granule representing R_i to M and, for each $1 \leq i \leq l$, we place an arrow from a module (or independent granule if*

[13] It should be emphasized that the **K** operator we use here has the flavor of syntactic sugar rather than of knowledge operators commonly employed in modal logics.

it exists) representing R'_i to M. In addition, we place an arrow from a database granule representing R to M.

6. *For each default granule we assign methods it will employ. Assume that the granule represents a default of the form $B_1 \wedge \cdots \wedge B_n : l/l$, where $n \geq 0$, and suppose that $R_1, \ldots, R_n, R$ are relation symbols occurring in $l_1, \ldots, l_n, l$, respectively. Assume further that $B'_1, \ldots, B'_k$ are those conjuncts of $B_1 \wedge \cdots \wedge B_n$ which are not preceded by $\mathbf{K}$ and let $R'_1, \ldots, R'_k$ be the relation symbols occurring in $B'_1, \ldots, B'_k$. First of all, there are two database methods, namely $A'.R^+$ and $A'.R^-$, where A' is the database granule responsible for the relation R. These methods are placed in the positive and the negative part of the diagram, respectively. In addition, there is one default method corresponding to the default represented by the granule. This method is placed in the positive part of the granule's diagram if l is positive and in the negative part otherwise. The default method is specified as follows.*
   a) *Suppose first that none of $R'_1, \ldots, R'_k$ is represented either by a default module or by an independent default granule. Then the default method used by the granule is given by*
   $$A_1.X_1 \wedge \cdots \wedge A_n.X_n \wedge A'.X, \tag{10.25}$$
   where $A_1, \ldots, A_n, A'$ are database granules responsible for $R_1, \ldots, R_n, R$, respectively, each X_i is ${R_i}^+$ (if l_i is positive) or ${R_i}^-$ (if l_i is negative) and X is $R^{\oplus}$ (if l is positive) or $R^{\ominus}$ (if l is negative).
   b) *Suppose now that at least one of $R'_1, \ldots, R'_k$ is represented by a default module or by an independent default granule. Let $R'_i, \ldots R'_j$ be all relation symbols from $R'_1, \ldots, R'_k$ that are represented by default modules (independent default granules) and suppose that $M_i, \ldots, M_j$ are the corresponding modules (independent default granules). Then the default method used by the granule is (10.25) with $A_i, \ldots, A_j$ replaced by $M_i, \ldots, M_j$, respectively.* ∎

The set of rules corresponding to a diagram of a normal default theory of Sp rough default logic is specified as usual. We also use the standard voting policy.

Theorem 10.3.3. *The set of rules R corresponding to a diagram $\mathcal{D}$ of a normal default theory of* Sp *default logic is stratified, provided that all prerequisities of all defaults are strong.* ∎

The notion of provability in a normal default theory and the notion of an extension of a normal default theory are specified as before (see Definitions 10.2.4 and 10.2.5).

Example 10.3.4. Suppose we are given the following:

John is a person.
Normally, a person is honest.

Normally, if a person is believed to be honest, she/he can be safely lent a small amount of money.

Normally, if a person is known to be honest, she/he can be safely lent a large amount of money.

In SP rough default logic these common-sense facts can be naturally represented by the following theory T:

$$W = \{P(\mathsf{j})\}$$

$$D = \left\{ \frac{P(x) : H(x)}{H(x)},\ \frac{H(x) : SM(x)}{SM(x)},\ \frac{\mathbf{K}H(x) : LM(x)}{LM(x)} \right\}.$$

where j, P, H, SM and LM stand for "John," "Person," "Honest," "Can be safely lent a small amount of money" and "Can be safely lent a large amount of money," respectively.

The CAKE diagram corresponding to the theory T is provided in Figure 10.10.

The set of rules corresponding to the diagram of the considered theory is the following.

$$A_1.P^+(\mathsf{j})$$
$$A_2.H^-(x) \rightarrow A_5.H^-(x)$$
$$\{A_2.H^+(x) \vee [A_1.P^+(x) \wedge \neg A_2.H^-(x)]\} \rightarrow A_5.H^+(x)$$
$$A_3.SM^-(x) \rightarrow A_6.SM^-(x)$$
$$\{A_3.SM^+(x) \vee [A_5.H^+(x) \wedge \neg A_3.SM^-(x)]\} \rightarrow A_6.SM^+(x)$$
$$A_4.LM^-(x) \rightarrow A_7.LM^-(x)$$
$$\{A_4.LM^+(x) \vee [A_2.H^+(x) \wedge \neg A_4.LM^-(x)]\} \rightarrow A_7.LM^+(x).$$

The partition P_1, P_2, where P_1 consists of the first rule and P_2 contains the remaining rules, provides a stratification of the above set.

The relations occurring in the heads of the above rules are defined by the fixpoint expression given by

$$\mathrm{LFP}\,\overline{X}.\mathcal{B}, \tag{10.26}$$

where $\overline{X}$ is the tuple $A_1.P^+$, $A_5.H^-$, $A_5.H^+$, $A_6.SM^-$, $A_6.SM^+$, $A_7.LM^-$, $A_7.LM^+$ and $\mathcal{B}$ is the conjunction of all the rules' bodies.

The computation of the relations specified by the expression (10.26) are provided below.

$$\{\}$$
$$\{A_1.P^+(\mathsf{j})\}$$
$$\{A_1.P^+(\mathsf{j}), A_5.H^+(\mathsf{j})\}$$
$$\{A_1.P^+(\mathsf{j}), A_5.P^+(\mathsf{j}), A_6.SM^+(\mathsf{j})\}.$$

The queries we are interested in are $SM(\mathsf{j})$ and $LM(\mathsf{j})$. It is immediately observed that, according to intuition, $SM(\mathsf{j})$ is in the extension of the theory T, whereas $LM(\mathsf{j})$ is not. ∎

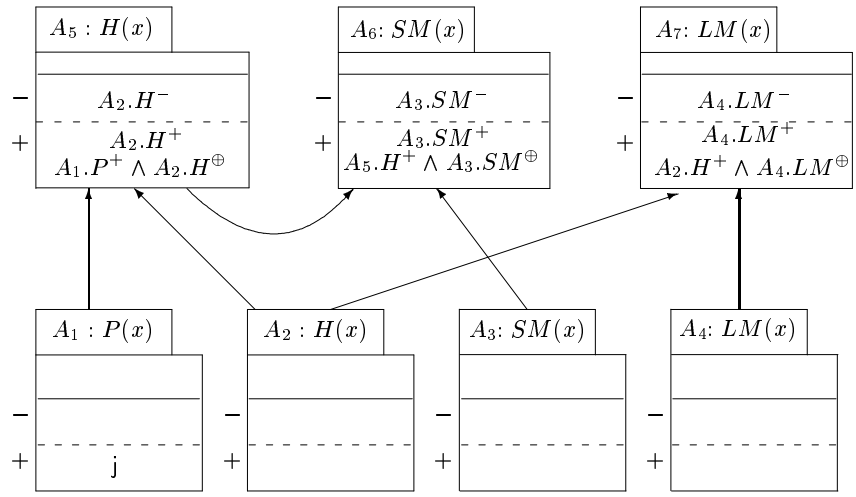

Fig. 10.10. Diagram corresponding to the theory of Example 10.3.4.

10.4 Prioritized Default Logic

Both rough default logic and Sp rough default logic can be easily extended to their prioritized versions (see Chapter 4, Section 4.2.6).

We start with some terminology, where $<$ is a strict partial order on a finite set X.

For all $x, y \in X$, y is said to be a *successor* of x if and only if $x < y$. With each $x \in X$ we associate a natural number, denoted by $depth(x)$, as follows:

$$depth(x) \stackrel{\text{def}}{=} \begin{cases} 1 + \max\{depth(x_1), \ldots, depth(x_k)\}, \\ \qquad \text{where } x_1, \ldots, x_k \in X \text{ are all successors of } x \\ 1 \quad \text{if } x \text{ has no successors.} \end{cases}$$

Intuitively, $depth(x)$ is the length of the longest sequence $x_0, x_1, \ldots, x_k$ such that $x_0 = x$ and $x_i < x_{i+1}$, for $i = 0, 1, \ldots, k-1$.

We define a sequence $X_1, \ldots X_n$ of subsets of X by

$$X_i = \{x \in X : depth(x) = i\}.$$

It is obvious that the sets $X_1, \ldots X_n$ form a partition of the set X. In the sequel, this partition will be referred to as *$<$-partition* of X.

Definition 10.4.1. *By a* normal prioritized theory of rough default logic *(respectively* Sp rough default logic*) we mean a triple $\langle W, D, < \rangle$, where $\langle W, D \rangle$ is a normal default theory of rough default logic (respectively* Sp *rough default logic) and $<$ is a strict partial order on D. Intuitively, $d_1 < d_2$ means that d_2 has higher priority than d_1.* ∎

The only difference between rough default logic (respectively SP rough default logic) and the prioritized version of rough default logic (respectively the prioritized version of SP rough default logic) lies in the use of a different voting policy.

Definition 10.4.2. *Let M be a default module associated with a relation symbol R and suppose that $d_1, \ldots, d_n$ are all the defaults of the considered theory containing R in their consequents. Assume further that $D_1, \ldots, D_k$ is a $<$-partition of the set $\{d_1, \ldots, d_n\}$.*

The modified voting process starts by applying the standard voting policy to all the granules representing the defaults from D_1. If the answer is TRUE *or* FALSE*, this is the final answer of the modified voting process. If the answer is* UNKNOWN *because at least one granule representing the defaults from D_1 answered* TRUE *and at least one of those granules answered* FALSE*, the final answer of the modified voting process is also* UNKNOWN*. Otherwise, i.e., if all the granules representing defaults from D_1 answered* UNKNOWN*, the standard voting policy is applied to the granules representing the defaults from D_2. This may continue for $D_3, \ldots, D_k$ unless at least one granule answers* TRUE *or* FALSE*.* ∎

Remark 10.4.3. Note that the voting policy can be modified in different ways if one is not satisfied with the policy described above. ∎

In order to specify diagrams for prioritized default logic theories, we will require the following additional notation.

Suppose that $A_1, \ldots, A_n$ are default granules responsible for a relation R. We write $V^+(A_1, \ldots, A_n; R)$, $V^-(A_1, \ldots, A_n; R)$ and $V^{\pm}(A_1, \ldots, A_n; R)$ as the abbreviations for the formulas:[14]

$$[A_1.R^+ \vee \cdots \vee A_n.R^+] \wedge \neg[A_1.R^- \vee \cdots \vee A_n.R^-]$$
$$[A_1.R^- \vee \cdots \vee A_n.R^-] \wedge \neg[A_1.R^+ \vee \cdots \vee A_n.R^+]$$
$$A_1.R^{\pm} \wedge \cdots \wedge A_n.R^{\pm}.$$

Definition 10.4.4. *Let $\langle W, D, <\rangle$ be a prioritized normal default theory of rough default logic (respectively* SP *rough default logic). The diagram corresponding to T is specified as before, see Definition 10.2.1 (respectively Definition 10.3.2) with the following additions.*

7. *Assume that granules $A_1, \ldots, A_n$, representing defaults $d_1, \ldots, d_n$, respectively, are grouped in a default module M. Let R be the relation symbol occurring in the consequents of $d_1, \ldots, d_n$ and let $D_1, \ldots, D_k$ be a $<$-partition of the set $\{d_1, \ldots, d_n\}$. If $k > 1$, then the adjudicating granule,*

[14] Observe that $V^+(A_1, \ldots, A_n; R)$ (respectively $V^-(A_1, \ldots, A_n; R)$) holds if the standard voting policy about R applied to the granules $A_1, \ldots, A_n$ returns the answer TRUE (respectively FALSE). The formula $V^{\pm}(A_1, \ldots, A_n; R)$ holds if and only if the answer of all the granules is UNKNOWN.

labeled by name : R, is introduced into the module M.[15] *As usual, name is a unique name of the introduced granule. For each* $1 \leq i \leq n$*, we put an arrow to the adjudicating granule from* A_i*.*

8. *Methods used by the new adjudicating granule are specified as follows. Assume that, for each* $1 \leq j \leq k$*,* $d_{j1}, d_{j2}, \ldots d_{ji_j}$ *are all the defaults from the set* D_j*. The following methods are put in the negative part of the diagram of the granule:*

$$V^-(A_{11}, \ldots, A_{1i_1}; R)$$
$$V^{\pm}(A_{11}, \ldots, A_{1i_1}; R) \wedge V^-(A_{21}, \ldots, A_{2i_2}; R)$$
$$\ldots$$
$$V^{\pm}(A_{11}, \ldots, A_{1i_1}; R) \wedge \cdots \wedge V^{\pm}(A_{(k-1)1}, \ldots, A_{(k-1)i_{k-1}}; R) \wedge V^-(A_{k1}, \ldots, A_{ki_k}; R).$$

The following methods are put in the positive part of the diagram of the granule:

$$V^+(A_{11}, \ldots, A_{1i_1}; R)$$
$$V^{\pm}(A_{11}, \ldots, A_{1i_1}; R) \wedge V^+(A_{21}, \ldots, A_{2i_2}; R)$$
$$\ldots$$
$$V^{\pm}(A_{11}, \ldots, A_{1i_1}; R) \wedge \cdots \wedge V^{\pm}(A_{(k-1)1}, \ldots, A_{(k-1)i_{k-1}}; R) \wedge V^+(A_{k1}, \ldots, A_{ki_k}; R).$$ ∎

The set of rules corresponding to a diagram of a prioritized normal default theory of rough default logic (SP rough default logic) is specified employing the usual CAKE methodology.[16]

Since the class of normal default theories is a subset of prioritized normal default theories, it is clear that prioritized normal default theories need not be stratified (see Example 10.2.2).

Provability in a normal default theory and the notion of an extension of a normal default theory are specified as before (see Definitions 10.2.4 and 10.2.5).

Example 10.4.5. Consider the theory

$$W = \{A(\mathrm{j}) \wedge FTS(\mathrm{j})\} \qquad D = \left\{ \frac{FTS(x) : \neg E(x)}{\neg E(x)}, \ \frac{A(x) : E(x)}{E(x)} \right\}.$$

The following is a non-prioritized default theory for Example 4.2.23 (Chapter 4, Section 4.2.6) with A, FTS, E and j standing for "Adult," "FullTimeStudent,", "Employed" and "John," respectively. The CAKE diagram corresponding to the theory T is provided in Figure 10.11.

[15] Note that $k > 1$ implies that there are priorities among the defaults represented in the module M.

[16] It should be emphasized that the presence of adjudicating granules changes the rules attached to modules (see Chapter 9, Section 9.5.3).

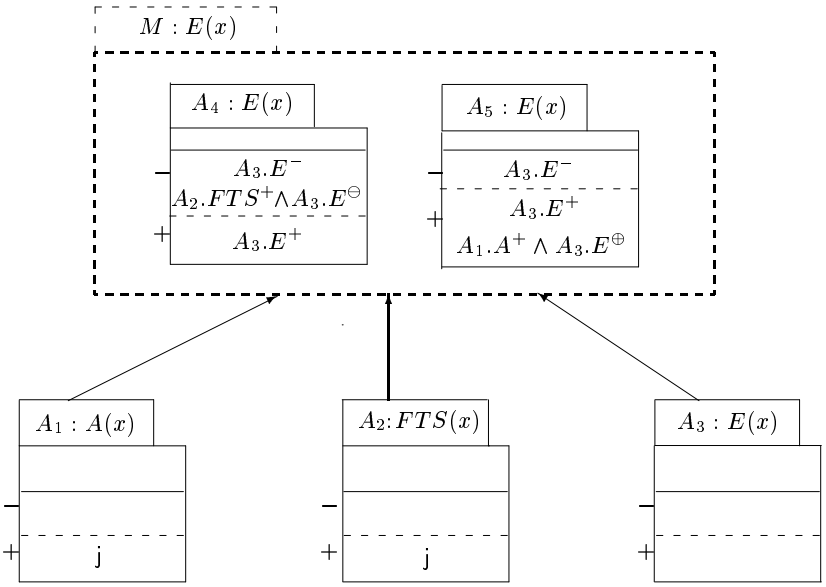

Fig. 10.11. CAKE diagram corresponding to the theory of Example 10.4.5.

Consider the query $E(\mathsf{j})$. It is easily verified that $E(\mathsf{j})$ is not a member of the extension of T. This is because this query is answered FALSE by the granule A_4 and TRUE by the granule A_5. Using the standard voting policy, the final answer is UNKNOWN. As we remarked earlier, see Example 4.2.23, this answer is intuitively problematic. Given that John is both a full time student and an adult the answer to the query should be FALSE. To achieve this effect, we can use a prioritized version of rough default logic with the following prioritization of rules: $\dfrac{A(x) : E(x)}{E(x)} < \dfrac{FTS(x) : \neg E(x)}{\neg E(x)}$.

The CAKE diagram for the prioritized theory is similar to that for the non-prioritized version, but in this case, the module M should instead be replaced by that from Figure 10.12. The following set of rules is associated with the diagram for the prioritized theory:

$A_1.A^+(\mathsf{j})$
$A_2.FTS^+(\mathsf{j})$
$\{A_3.E^-(x) \vee [A_2.FTS^+(x) \wedge \neg A_3.E^+(x)]\} \rightarrow A_4.E^-(x)$
$A_3.E^+(x) \rightarrow A_4.E^+(x)$
$A_3.E^-(x) \rightarrow A_5.E^-(x)$
$\{A_3.E^+(x) \vee [A_1.A^+(x) \wedge \neg A_3.E^-(x)]\} \rightarrow A_5.E^+(x)$
$\{A_4.E^-(x) \vee [\neg A_4.E^+(x) \wedge \neg A_4.E^-(x) \wedge A_5.E^-(x)]\} \rightarrow A_6.E^-(x)$

$\{A_4.E^+(x) \vee [\neg A_4.E^+(x) \wedge \neg A_4.E^-(x) \wedge A_5.E^+(x)]\} \rightarrow A_6.E^+(x)$
$A_6.E^+(x) \rightarrow M.E^+(x)$
$A_6.E^-(x) \rightarrow M.E^-(x).$

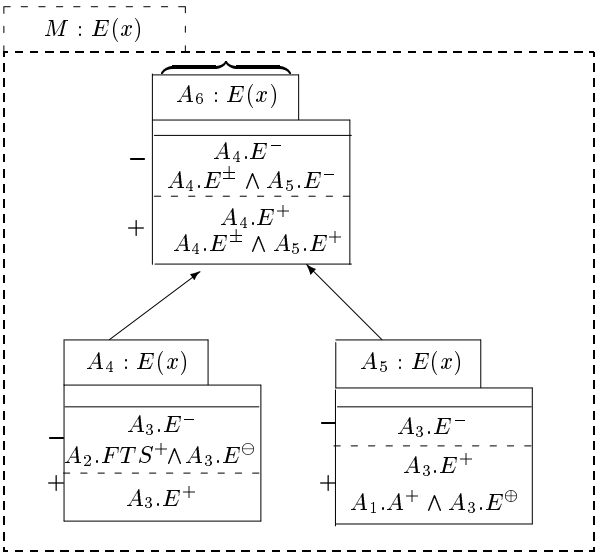

Fig. 10.12. New module corresponding to the prioritized theory of Example 10.4.5.

The above set is clearly stratified.

The relations occurring in the heads of the above rules are specified by the fixpoint expression given by

$$\mathrm{LFP}\,\overline{X}.\mathcal{B}, \tag{10.27}$$

where $\overline{X}$ is the tuple $A_1.A^+$, $A_2.FTS^+$, $A_4.E^+$, $A_4.E^-$, $A_5.E^+$, $A_5.E^-$, $A_6.E^+$, $A_6.E^-$, $M.E^+$, $M.E^-$ and $\mathcal{B}$ is the conjunction of all the rules' bodies.

Computing the relations defined by (10.27), results in the following:

$\{\}$
$\{A_1.A^+(\mathrm{j}), A_2.FTS^+(\mathrm{j})\}$
$\{A_1.A^+(\mathrm{j}), A_2.FTS^+(\mathrm{j}), A_4.E^-(\mathrm{j}), A_5.E^+(\mathrm{j})\}$
$\{A_1.A^+(\mathrm{j}), A_2.FTS^+(\mathrm{j}), A_4.E^-(\mathrm{j}), A_5.E^+(\mathrm{j}), A_6.E^-(\mathrm{j})\}$
$\{A_1.A^+(\mathrm{j}), A_2.FTS^+(\mathrm{j}), A_4.E^-(\mathrm{j}), A_5.E^+(\mathrm{j}), A_6.E^-(\mathrm{j}), M.E^-(\mathrm{j})\}.$

Since j satisfies the $M.E^-$-coordinate of (10.27), we conclude that $\neg E(\mathrm{j})$ belongs to the extension of the prioritized theory. ∎

It is obvious that analogous theorems to Theorems 10.2.7 and 10.2.8 hold for prioritized versions of rough default logic and Sp rough default logic.

10.5 Semi-Normal Default Theories

In this section, we investigate disjunction-free semi-normal non-prioritized default theories of rough default logic.[17]

Recall that a disjunction-free semi-normal default is any default of the form

$$\frac{l_1 \wedge \cdots \wedge l_n : l'_1 \wedge \cdots \wedge l'_k \wedge l}{l}$$

where $l_1, \ldots, l_n, l'_1, \ldots, l'_k, l$ are literals.

To deal with semi-normal default theories, we proceed analogously as before. The only difference is that additional conjuncts occurring in justifications of semi-normal defaults have to be taken under consideration. The next example illustrates the use of semi-normal defaults.

Example 10.5.1. Consider theory $T = \langle W, D \rangle$, where

$$W = \{D(\mathsf{b})\} \text{ and } D = \left\{ \frac{D(x) : A(x)}{A(x)}, \ \frac{A(x) : \neg D(x) \wedge E(x)}{E(x)} \right\}.$$

Here A, D, E, and b stand for "Adult," "HighSchoolDropout," "Employed" and "Bill," respectively (see Example 4.2.17, Chapter 4, Section 4.2.4.)

The Cake diagram corresponding to the theory T is provided in Figure 10.13.

Consider the granule A_5 representing the semi-normal default of our theory. Note that its default method refers to the relation D which occurs in the justification of the default.

The following is the set of rules associated with the above diagram.

$A_2.D^+(\mathsf{b})$
$A_1.A^-(x) \rightarrow A_4.A^-(x)$
$\{A_1.A^+(x) \vee [A_2.D^+(x) \wedge \neg A_1.A^-(x)]\} \rightarrow A_4.A^+(x)$
$A_3.E^-(x) \rightarrow A_5.E^-(x)$
$\{A_3.E^+(x) \vee [A_4.A^+(x) \wedge \neg A_2.D^+(x) \wedge \neg A_3.E^-(x)]\} \rightarrow A_5.E^+(x).$

The partition P_1, P_2, where P_1 consists of the first rule and P_2 contains the remaining rules, provides a stratification of the above set.

The fixpoint expression specifying the relations occurring in the heads of the rules is given by

$$\text{LFP}\, A_2.D^+, A_4.A^-, A_4.A^+, A_5.E^-, A_5.E^+.\mathcal{B}, \tag{10.28}$$

[17] The results in this section can easily be generalized to Sp rough default logic and to prioritized versions of both rough default logic and Sp rough default logic.

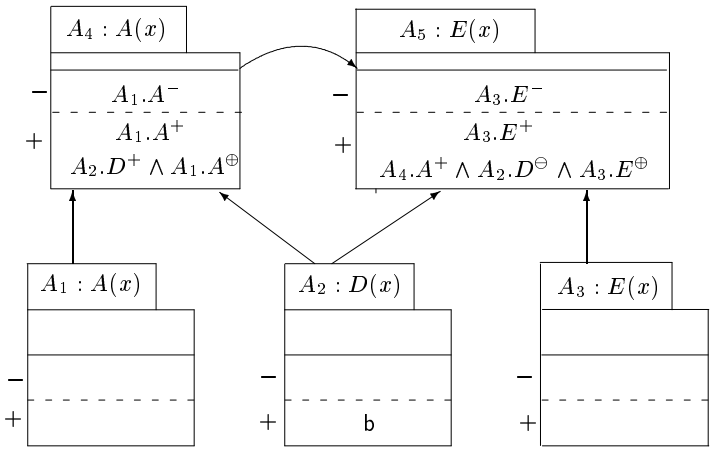

Fig. 10.13. Diagram corresponding to the theory of Example 10.5.1.

where $\mathcal{B}$ denotes the conjunction of the rules' bodies.

The resulting computation of the relations specified is provided below. (10.28).

$\{\}$
$\{A_2.D^+(\mathsf{b})\}$
$\{A_2.D^+(\mathsf{b}), A_4.A^+(\mathsf{b})\}$.

Suppose that the query of interest is $E(\mathsf{b})$. It is immediately observed that, according to expectations, b neither satisfies the $A_5.E^+$-coordinate nor the $A_5.E^-$-coordinate of the fixpoint expression (10.28). Consequently, the answer to the query $E(\mathsf{b})$ is UNKNOWN. ∎

In dealing with semi-normal default theories, the construction of CAKE diagrams has to be adjusted.

Definition 10.5.2. *Let T be a disjunction-free semi-normal default theory. A* CAKE *diagram corresponding to the theory T is constructed in a manner similar to that from Definition 10.2.1 with the exception that points 4, 5 and 6 are replaced by the following:*

4. *Let A be an independent default granule representing a default d and suppose that the consequent of d contains a relation symbol R. Assume further that $R_1, \ldots, R_k$ are all the relation symbols occurring in the prerequisite of d and let $R_{k+1}, \ldots, R_m$ be all the relation symbols, except R, occurring in the justification of d. For each $1 \leq i \leq m$, we place an arrow from a database granule representing R_i to A and from a module (or independent default granule if it exists) representing R_i to A. An arrow is also placed from a database granule representing R to A.*

5. *Let M be a module representing a relation R. Let $A_1, \ldots, A_p$ be default granules of M and suppose that A_i corresponds to a default d_i. Assume further that $R_1, \ldots, R_k$ are all the relation symbols occurring in the prerequisites of $d_1, \ldots, d_p$ and let $R_{k+1}, \ldots, R_m$ be all the relation symbols, except R, occurring in the justifications of $d_1, \ldots, d_p$. For each $1 \leq i \leq m$, we place an arrow from a database granule representing R_i to M and, from a module (or independent default granule if it exists) representing R_i to M. In addition, we place an arrow from a database granule representing R to M.*
6. *For each default granule (dependent or not) we now assign methods it will employ. Assume that the granule represents a default of the form $l_1 \wedge \cdots \wedge l_n : l_{n+1} \wedge \cdots \wedge l_m \wedge l/l$, where $n \geq 0$, and suppose that $R_1, \ldots, R_n, R_{n+1}, \ldots, R_m$, R are the relation symbols occurring in $l_1, \ldots, l_n, l_{n+1}, \ldots, l_m, l$, respectively. First of all, there are two methods, namely $A_k.R^+$ and $A_k.R^-$, where A_k is the database granule responsible for the relation R. These methods are placed in the positive and negative parts of the diagram, respectively. In addition, there is one method corresponding to the default represented by the granule. This method is placed in the positive part of the granule's diagram if l is a positive literal and in the negative part otherwise. The default method is constructed as follows.*
   a) *Suppose first that none of $R_1, \ldots, R_n, R_{n+1}, \ldots, R_m$ is represented either by a default module or by an independent default granule. Then the default method is given by*
   $$A_1.X_1 \wedge \cdots \wedge A_n.X_n \wedge A_{n+1}.X_{n+1} \wedge \cdots \wedge A_m.X_m \wedge A.X \quad (10.29)$$
   where $A_1, \ldots, A_n, A_{n+1}, \ldots, A_m, A$ are database granules responsible for $R_1, \ldots, R_n, R_{n+1}, \ldots, R_m, R$, respectively, each X_i ($1 \leq i \leq n$) is ${R_i}^+$, if l_i is positive, or ${R_i}^-$, if l_i is negative, each X_i ($n+1 \leq i \leq m$) is $R_i^{\oplus}$, if l_i is positive, or $R_i^{\ominus}$, if l_i is negative. Finally, X is $R^{\oplus}$, if l is positive, or $R^{\ominus}$, if l is negative.
   b) *Suppose now that at least one of $R_1, \ldots, R_n, R_{n+1}, \ldots, R_m$ is represented by a default module or by an independent default granule. Let $R_i, \ldots R_j$ be all relations from $R_1, \ldots, R_n, R_{n+1}, \ldots, R_m$ that are represented by default modules (independent default granules) and suppose that $M_i, \ldots, M_j$ are the corresponding modules (independent default granules). Then the default method is (10.29) with $A_i, \ldots, A_j$ replaced by $M_i, \ldots, M_j$, respectively.* ∎

Since the class of normal default theories is a subset of semi-normal default theories, it is clear that semi-normal default theories need not be stratified (see Example 10.2.2).

For stratified semi-normal default theories the notion of an extension is defined as usual. The Theorems 10.2.7 and 10.2.8 also hold for stratified semi-normal default theories.

10.6 Non-stratified Default Theories

The notion of an extension of a default theory have been defined for stratified theories only. However, as we have seen (Example 10.2.2), even normal theories need not be stratified. In this section, we briefly describe how non-stratified theories can by approached using the well-founded semantics (see Section 3.5.2).

The process of computing an extension of a default theory T using the well-founded semantics proceeds in three steps.

1. we replace a set of CAKE rules assigned to the diagram of the theory T by a corresponding DATALOG$^\neg$ program P_T
2. we compute the well-founded model of P_T
3. using the model obtained in step 2, we compute an extension of T.

The steps 1. and 2. above, have been described in Sections 3.5.2 and 9.5.5, respectively. To describe step 3, we have to reformulate the notion of provability from Definition 10.2.4.

Definition 10.6.1. *Let T be a default theory and suppose that* $\mathbf{R}$ *is the set of rules corresponding to the diagram of T. Suppose further that I is the well-founded model of the* DATALOG$^\neg$ *program corresponding to* $\mathbf{R}$*. A ground literal of the form $P(\overline{c})$ (respectively $\neg P(\overline{c})$) is said to be* wf-provable *in T if and only if $A.P^+$ (respectively $A.P^-$) is true in I and A is the name associated with a database granule, an independent default granule or a default module.* ∎

Definition 10.6.2. *Let T be a default theory. An* wf-extension *of T is the set of all ground literals which are provable in T.* ∎

Example 10.6.3 (Example 10.2.2 continued). The program corresponding to the rules (10.10)-(10.22) is the following.

$$A_3.Q^+(\mathsf{n}) \leftarrow$$
$$A_5.P^-(x) \leftarrow A_2.P^-(x)$$
$$A_5.P^-(x) \leftarrow A_1.R^+(x), \neg A_2.P^+(x)$$
$$A_5.P^+(x) \leftarrow A_2.P^+(x)$$
$$A_6.P^-(x) \leftarrow A_2.P^-(x)$$
$$A_6.P^+(x) \leftarrow A_2.P^+(x)$$
$$A_6.P^+(x) \leftarrow N.Q^+(x), \neg A_2.P^-(x)$$
$$A_7.Q^-(x) \leftarrow A_3.Q^-(x)$$
$$A_7.Q^+(x) \leftarrow A_3.Q^+(x)$$
$$A_7.Q^+(x) \leftarrow M.P^+(x), \neg A_3.Q^-(x)$$
$$A_8.Q^-(x) \leftarrow A_3.Q^-(x)$$
$$A_8.Q^+(x) \leftarrow A_3.Q^+(x)$$

$$A_8.Q^+(x) \leftarrow A_4.S^+(x), \neg A_3.Q^-(x)$$
$$M.P^-(x) \leftarrow A_5.P^-(x), \neg A_5.P^+(x), \neg A_6.P^+(x)$$
$$M.P^-(x) \leftarrow A_6.P^-(x), \neg A_5.P^+(x), \neg A_6.P^+(x)$$
$$M.P^+(x) \leftarrow A_5.P^+(x), \neg A_5.P^-(x), \neg A_6.P^-(x)$$
$$M.P^+(x) \leftarrow A_6.P^+(x), \neg A_5.P^-(x), \neg A_6.P^-(x)$$
$$N.Q^-(x) \leftarrow A_7.Q^-(x), \neg A_7.Q^+(x), \neg A_8.Q^+(x)$$
$$N.Q^-(x) \leftarrow A_8.Q^-(x), \neg A_7.Q^+(x), \neg A_8.Q^+(x)$$
$$N.Q^+(x) \leftarrow A_7.Q^+(x), \neg A_7.Q^-(x), \neg A_8.Q^-(x)$$
$$N.Q^+(x) \leftarrow A_8.Q^+(x), \neg A_7.Q^-(x), \neg A_8.Q^-(x).$$

Using the method described in Section 3.5.2, it is easily checked that the well-founded model of the above program is given by

$$\{\ N.Q^+(\mathsf{n}), A_3.Q^+(\mathsf{n}), A_7.Q^+(\mathsf{n}), A_8.Q^+(\mathsf{n}),$$
$$\neg N.Q^-(\mathsf{n}), \neg A_3 Q^-(\mathsf{n}), \neg A_7.Q^-(\mathsf{n}), \neg A_8.Q^-(\mathsf{n}),$$
$$M.P^+(\mathsf{n}), A_6.P^+(\mathsf{n}),$$
$$\neg M.P^-(\mathsf{n}), \neg A_2.P^-(\mathsf{n}), \neg A_2.P^+(\mathsf{n}),$$
$$\neg A_5.P^-(\mathsf{n}), \neg A_5.P^+(\mathsf{n}), \neg A_6.P^-(\mathsf{n}),$$
$$\neg A_1.R^+(\mathsf{n}),$$
$$\neg A_4.S^+(\mathsf{n})\ \}.$$

Thus, the theory T has a wf-extension given by

$$E = \{P(\mathsf{n}), Q(\mathsf{n})\}.$$

We conclude this section by remarking that for stratified default theories extensions and wf-extensions are identical.

10.7 Conclusions

In this chapter, we have shown that a subset of default logic can easily be formalized using the CAKE methodology. The main advantages of this approach are that the computation of an extension of a default theory is tractable and the extensions are intuitively sound for many of the problematic examples previously considered in the literature. As shown for Reiter's default logic in [80], this problem is computationally hard in the general case, even for propositional disjunction-free theories.

The use of rough set ideas has provided us with a straightforward way to distinguish between "hard" and "soft" facts, when considering the prerequisite of a default. This allows us to properly model various naturally occurring scenarios that are very difficult to model intuitively in traditional versions of default logic.

General non-normal default theories have not been considered in this chapter. In our opinion, they are less interesting from a practical point of view. As we argued elsewhere, all naturally occurring scenarios can be properly dealt with in the framework of semi-normal default theories (for references see Section 10.8).

To compute an extension of a stratified default theory, we construct an appropriate "global" fixpoint expression. In practice, we are generally interested in single queries. For this reason, it would be reasonable to specify a top-down algorithm which performs only necessary calculations. Defining such an algorithm is straightforward, provided that a CAKE diagram corresponding to a given theory does not contain any cycles. To specify such an algorithm in the general case is an interesting research problem.

10.8 Bibliographic Notes

The version of default logic we have presented here is quite original. It has been reported in [46]. Of course, our considerations have been heavily influenced by the general literature concerning default logic. For references see Section 4.4, Chapter 4.

The idea of distinguishing between weak and strong prerequisites originates from [39] (see also [143]).

The observation that all naturally occurring scenarios can be properly dealt with in the framework of semi-normal default theories has been made in [95].