Exercises-3 (Circumscription)

Definition 1 (Predicate Circumscription) The *circumscription schema* of $P_1, \ldots, P_n$ in a theory T, denoted by $\mathrm{CS}(T; P_1, \ldots, P_n)$, is the second-order formula

$$\left[T(\Phi_1, \ldots, \Phi_n) \wedge \bigwedge_{i=1}^{n} [\Phi_i \leq P_i]\right] \supset \bigwedge_{i=1}^{n} [P_i \leq \Phi_i]$$

or, equivalently

$$\left[T(\Phi_1, \ldots, \Phi_n) \wedge \bigwedge_{i=1}^{n} [\forall \overline{x}.\ \Phi_i(\overline{x}) \supset P_i(\overline{x})]\right] \supset \bigwedge_{i=1}^{n} [\forall \overline{x}.\ P_i(\overline{x}) \supset \Phi_i(\overline{x})].$$

Here $\Phi_1, \ldots, \Phi_n$ are predicate variables with the same arities as $P_1, \ldots, P_n$, respectively. $T(\Phi_1, \ldots, \Phi_n)$ is the sentence obtained from $T(P_1, \ldots, P_n)$ by replacing all occurrences of $P_1, \ldots, P_n$ by $\Phi_1, \ldots, \Phi_n$, respectively.

Let $\overline{\mathrm{CS}}(T; P_1, \ldots, P_n)$ be the set of all formulas obtainable from $\mathrm{CS}(T; P_1, \ldots, P_n)$ by proper substitutions of first-order predicate expressions for $\Phi_1, \ldots, \Phi_n$. The *predicate circumscription* of $\overline{P} = (P_1, \ldots, P_n)$ in T, written $\mathrm{CIRC}_{\mathrm{PR}}(T; \overline{P})$, is $T \cup \overline{\mathrm{CS}}(T; P_1, \ldots, P_n)$.

The set of theorems derivable by circumscribing $\overline{P} = (P_1, \ldots, P_n)$ in T is identified with the set of theorems classically entailed by $\mathrm{CIRC}_{\mathrm{PR}}(T; \overline{P})$.

Definition 2 (Second-Order Circumscription) Let $\overline{P}$ be a tuple of distinct predicate constants, let $\overline{S}$ be a tuple of distinct function and/or predicate constants disjoint with $\overline{P}$, and suppose that $T(\overline{P}, \overline{S})$ is a first- or second-order theory. The *second-order circumscription of* $\overline{P}$ in T *with variable* $\overline{S}$, written $\mathrm{CIRC}_{\mathrm{SO}}(T; \overline{P}; \overline{S})$, is the sentence

$$T(\overline{P}, \overline{S}) \wedge \forall \overline{\Phi}\overline{\Psi}\ \neg[T(\overline{\Phi}, \overline{\Psi}) \wedge \overline{\Phi} < \overline{P}]$$

where $\overline{\Phi}$ and $\overline{\Psi}$ are tuples of variables similar to $\overline{P}$ and $\overline{S}$, respectively.

$\mathrm{CIRC}_{\mathrm{SO}}(T; \overline{P}; \overline{S})$ can be rewritten as

$$T(\overline{P}, \overline{S}) \wedge \forall \overline{\Phi}\overline{\Psi}\ \left(T(\overline{\Phi}, \overline{\Psi}) \wedge (\overline{\Phi} \leq \overline{P}) \supset (\overline{P} \leq \overline{\Phi})\right)$$

which, in turn, is an abbreviation for

$$T(\overline{P}, \overline{S}) \wedge \forall \overline{\Phi}\overline{\Psi} \left[\left(T(\overline{\Phi}, \overline{\Psi}) \wedge \bigwedge_{i=1}^{n} [\forall \overline{x}.\ \Phi_i(\overline{x}) \supset P_i(\overline{x})]\right) \supset \bigwedge_{i=1}^{n} [\forall \overline{x}.\ P_i(\overline{x}) \supset \Phi_i(\overline{x})]\right].$$

A formula A is said to be a *consequence* of second-order circumscription of $\overline{P}$ in T with *variable* $\overline{S}$ iff $\mathrm{CIRC}_{\mathrm{SO}}(T; \overline{P}; \overline{S}) \models A$.

1. Let $T = \{Block(a) \vee Block(b)\}$. $\quad \overline{P} = (Block)$.

   Show that $\mathrm{CIRC}_{\mathrm{PR}}(T; \overline{P}) \models \forall x.\, [Block(x) \supset x = a] \vee \forall x.\, [Block(x) \supset x = b]$.

2. Let $T = \{Nat(0) \wedge \forall x[Nat(x) \supset Nat(succ(x)]\}$.

   Show that using predicate circumscription, with second-order substitutions, it is possible to obtain from T the usual axiom schema of induction, i.e. the formula

   $$\Big[\Psi(0) \wedge \forall x[\Psi(x) \wedge Nat(x) \supset \Psi(succ(x))]\Big] \supset \forall x[Nat(x) \supset \Psi(x)].$$

3. Let T consist of:

   $Bird(Tweety)$
   $\forall x.\, Penguin(x) \supset Bird(x) \wedge \neg Flies(x)$
   $\forall x.\, Bird(x) \wedge \neg Ab(x) \supset Flies(x)$.

   Show that using second-order circumscription, it is possible to infer

   $$Flies(Tweety) \wedge \forall x \neg Penguin(x)$$

   from T.

4. Assume that the following facts are true:

   A person normally lives in the same city as his wife.
   Bill's wife lives in Vancouver.

   This can be represented by the theory T:

   $\forall x.\, \neg Ab(x) \supset city(x) = city(wife(x))$
   $city(wife(Bill)) = Vancouver$.

   Show that using second-order circumscription, it is possible to infer

   $$city(Bill) = Vancouver$$

   from T.

5. Let $T = \{P(a) \wedge P(b) \wedge Q(b) \wedge [Q(a) \supset P(c)]\}$.

   Show that using second-order circumscription, with P minimized and a, c allowed to vary, it is possible to derive $(a = b) \wedge Q(a) \wedge P(c)$.

Solutions

1. $T = \{Block(a) \vee Block(b)\}$. $\quad \overline{P} = (Block)$.

   $\mathrm{CS}(T;Block) = \Big[[\Phi(a) \vee \Phi(b)] \wedge \forall x[\Phi(x) \supset Block(x)]\Big] \supset \forall x[Block(x) \supset \Phi(x)]$.

   Taking the substitutions

   $$\{\Phi \leftarrow \lambda x.x = a\} \quad \text{and} \quad \{\Phi \leftarrow \lambda x.x = b\}$$

   we successively obtain

   $$\Big[[a = a \vee a = b] \wedge \forall x[x = a \supset Block(x)]\Big] \supset \forall x[Block(x) \supset x = a]$$

   which simplifies to

   $$(1) \quad Block(a) \supset \forall x[Block(x) \supset x = a]$$

   and

   $$\Big[[b = a \vee b = b] \wedge \forall x[x = b \supset Block(x)]\Big] \supset \forall x[Block(x) \supset x = b]$$

   which simplifies to

   $$(2) \quad Block(b) \supset \forall x[Block(x) \supset x = b].$$

   (1), (2) and T yield

   $$\mathrm{CIRC}_{\mathrm{PR}}(T;\overline{P}) \models \forall x.\, [Block(x) \supset x = a] \vee \forall x.\, [Block(x) \supset x = b].$$

2. $T = \{Nat(0) \wedge \forall x[Nat(x) \supset Nat(succ(x))]\}$. $\quad \overline{P}=(Nat)$.

   $$\mathrm{CS}(T,\overline{P}) = \Big[\Phi(0) \wedge \forall x[\Phi(x) \supset \Phi(succ(x))] \wedge \forall x[\Phi(x) \supset Nat(x)]\Big] \supset \supset \forall x[Nat(x) \supset \Phi(x)].$$

   Applying the substitution $\{\Phi \leftarrow \lambda x[\Psi(x) \wedge Nat(x)]\}$, we obtain

   $$\Big[[\Psi(0) \wedge Nat(0)] \wedge \forall x[\Psi(x) \wedge Nat(x) \supset \Psi(succ(x)) \wedge Nat(succ(x))] \wedge \wedge \forall x[\Psi(x) \wedge Nat(x) \supset Nat(x)]\Big] \supset \forall x[Nat(x) \supset \Psi(x) \wedge Nat(x)]$$

   which simplifies to

   $$\Big[[\Psi(0) \wedge Nat(0)] \wedge \forall x[\Psi(x) \wedge Nat(x) \supset \Psi(succ(x)) \wedge Nat(succ(x))]\Big] \supset \supset \forall x[Nat(x) \supset \Psi(x)].$$

   This, together with T, yield

   $$\mathrm{CIRC}_{\mathrm{PR}}(T;\overline{P}) \models \Big[\Psi(0) \wedge \forall x[\Psi(x) \wedge Nat(x) \supset \Psi(succ(x))]\Big] \supset \supset \forall x[Nat(x) \supset \Psi(x)].$$

3. T consists of:

$$Bird(Tweety)$$
$$\forall x.\ Penguin(x) \supset Bird(x) \wedge \neg Flies(x)$$
$$\forall x.\ Bird(x) \wedge \neg Ab(x) \supset Flies(x).$$

We minimize $\overline{P}$=(Ab) with $\overline{S}$= ($Flies, Penguin$) allowed to vary.

$\mathrm{CIRC}_{\mathrm{SO}}(T; \overline{P}; \overline{S})$=

$$T(\overline{P}, \overline{S}) \wedge \forall \Phi \Psi_1 \Psi_2 \left\{ \left[[\forall x.\ \Psi_2(x) \supset Bird(x) \wedge \neg \Psi_1(x)] \wedge \right.\right.$$
$$\wedge [\forall x.\ Bird(x) \wedge \neg \Phi(x) \supset \Psi_1(x)] \wedge$$
$$\left.\left. \wedge [\forall x.\ \Phi(x) \supset Ab(x)] \right] \supset [\forall x.\ Ab(x) \supset \Phi(x)] \right\}.$$

Applying the substitution $\{\Phi \leftarrow \lambda x.False,\ \Psi_1 \leftarrow \lambda x.True,\ \Psi_2 \leftarrow \lambda x.\ False\}$, we get

$$T(\overline{P}, \overline{S}) \wedge \left\{ \left[[\forall x.\ False \supset Bird(x) \wedge False] \wedge \right.\right.$$
$$\wedge [\forall x.\ Bird(x) \wedge True \supset True] \wedge$$
$$\left.\left. \wedge [\forall x.\ False \supset Ab(x)] \right] \supset [\forall x.\ Ab(x) \supset False] \right\}$$

which simplifies to

$$T(\overline{P}, \overline{S}) \wedge \forall x\ \neg Ab(x).$$

From this, we immediately obtain

$$\mathrm{CIRC}_{\mathrm{SO}}(T; \overline{P}; \overline{S}) \models Flies(Tweety) \wedge \forall x \neg Pinguin(x).$$

Taking $Penguin$ as varying predicate may seem unreasonable. However, if we restrict $\overline{S}$ to $Flies$, all we can be able to infer is that birds fly, except penguins. In particular, since we have no information that Tweety is not a penguin, we shall not be able to infer that Tweety flies.

4. $T = \{[\forall x.\ \neg Ab(x) \supset city(x) = city(wife(x))] \wedge city(wife(Bill)) = Vancouver\}$.

We minimize $\overline{P} = (Ab)$ with $\overline{S} = (city)$ allowed to vary.

$\mathrm{CIRC}_{\mathrm{SO}}(T; \overline{P}; \overline{S})$=

$$T(\overline{P}, \overline{S}) \wedge \forall \Phi \psi \left\{ \left[[\forall x.\ \neg \Phi(x) \supset \psi(x) = \psi(wife(x))] \wedge \right.\right.$$
$$\left.\left. \wedge [\psi(wife(Bill)) = Vancouver] \wedge [\forall x.\ \Phi(x) \supset Ab(x)] \right] \supset \right.$$
$$\left. \supset [\forall x.\ Ab(x) \supset \Phi(x)] \right\}.$$

Taking the substitution[1] $\{\Phi \leftarrow \lambda x.\ False,\ \psi \leftarrow \lambda x.\ Vancouver\}$, we get

$$T(\overline{P},\overline{S}) \wedge \Big\{ \Big[[\forall x.\ True \supset Vancouver = Vancouver] \wedge \\ \wedge [Vancouver = Vancouver] \wedge [\forall x.\ False \supset Ab(x)] \Big] \supset \\ \supset [\forall x.\ Ab(x) \supset False] \Big\}$$

which simplifies to

$$T(\overline{P},\overline{S}) \wedge \forall x\ \neg Ab(x).$$

From this, we immediately obtain

$$\mathrm{CIRC}_{\mathrm{SO}}(T;\overline{P};\overline{S}) \models city(Bill) = Vancouver.$$

5. $T = \{P(a) \wedge P(b) \wedge Q(b) \wedge [Q(a) \supset P(c)]\}$. $\overline{P}$=(P); $\overline{S}$=(a,c).

$\mathrm{CIRC}_{\mathrm{SO}}(T;\overline{P};\overline{S})$=

$$T(\overline{P},\overline{S}) \wedge \forall\Phi\forall yz \Big\{ \Big[[\Phi(y) \wedge \Phi(b) \wedge Q(b)] \wedge \\ \wedge [Q(y) \supset \Phi(z)] \wedge [\forall x.\ \Phi(x) \supset P(x)] \Big] \supset \\ \supset [\forall x.\ P(x) \supset \Phi(x)] \Big\}.$$

Taking the substitution $\{\Phi \leftarrow \lambda x.\ x = b,\ y \leftarrow \lambda x.\ b,\ z \leftarrow \lambda x.\ b\}$ and simplifying, we obtain

$$\mathrm{CIRC}_{\mathrm{SO}}(T;\overline{P};\overline{S}) \models \forall x.\ P(x) \supset x = b$$

from which we get

$$\mathrm{CIRC}_{\mathrm{SO}}(T;\overline{P};\overline{S}) \models (a = b) \wedge Q(a) \wedge P(c).$$

NOTE: If function constants are allowed to vary, then second-order circumscription may serve to obtain

- new ground equality statements
- positive instances of circumscribed predicates
- new instances of non-variable predicates.

These types of conclusions cannot be obtained by using second-order circumscription if only predicates are allowed to vary.

[1] Observe that we substitute here a predicate expression for a predicate variable and a function expression for a function variable.