FOUNDATIONS OF N0N-MONOTONIC

REASONING

Monotonicity:

For any sets of premisses S and S':
$S \subseteq S'$ implies $\{A : S \vdash A\} \subseteq \{A : S' \vdash A\}$.

(F1) On Saturday evenings John usually visits his club.

(F2) It is Saturday evening.

(C) John is at the club.

(F3) Yesterday John had a car accident.

Definition 1
By *non-monotonic reasoning* we understand the drawing of conclusions which may be invalidated in the light of new information. A logical system is called *non-monotonic* iff its provability relation violates the property of monotonicity.

Non-monotonic reasoning has been primarily studied in the context of common-sense reasoning.

- formal logical reasoning – based on the concept of *truth*
- common-sense inference – grounded on the concept of *rationality*.

We cannot expect that our common-sense conclusions will be always true. However, we always demand them to be rational.

Rationality enjoys two specific properties:

- It is *agent-dependent*: different agents may be of different opinions as to what is rational in a given situation.
- It is *purpose-dependent*: the acceptance of a proposition as a rational conclusion depends on the purpose it is to be used for. For instance, given cursory evidence, I may well assume that Bill is honest and lend him 100 crowns. But I will be more catious, and try to conduct an investigation, if I was to consider him as my business partner.

To stress the subjective nature of common-sense conclusions, they are often called *beliefs* in the AI literature.

Two important questions:

- What really are common-sense conclusions (beliefs)?
- What techniques do we use when reaching them?

Definition 2 (Perlis) A proposition p is believed by an agent g, i.e. g views p as a rational conclusion, if g is prepared to use p as if it were true.

(F1) On Saturday evenings John usually visits his club.

(F2) It is Saturday evening.

(C) John is at the club.

Is (C) based on the rule

(R) From (F1) and (F2) infer (C)?

No, it is grounded on the rule

(R') From (F1) and (F2), in the absence of evidence to suspect otherwise, infer (C).

Definition 3 By a *non-monotonic inference pattern* (a *non-monotonic rule*) we understand the following reasoning schema:

"Given information A, in the absence of evidence B, infer a conclusion C."

An important case:

"Given information A, in the absence of evidence to the contrary, infer C."

Non-monotonic inference involves two problems:

Problem of formalization: To determine how our knowledge is to be formalized. This includes both specifying how this knowledge is to be represented and defining the set of conclusions (beliefs) derivable from such a representation.

Belief-revision problem: To update the current model of the world when new evidence invalidates some currently accepted beliefs.

Non-monotonic reasoning and reasoning about action

Frame problem: The problem of representing which properties persist and which properties change when actions are performed.

Corresponding non-monotonic rule: "All aspects of the world remain invariant except for those explicitely changed."

Qualification problem: How to represent all qualifications which should be satisfied to assure the success of the performed action.

Corresponding non-monotonic rule: "In the absence of evidence to the contrary, assume that an action succeeds."

Default logic (Reiter)

Default:

$$\frac{Bird(x) : \ Flies(x)}{Flies(x)}$$

Default theory:

Axioms + defaults

Definition A *default* is any expression of the form

$$\frac{A(\overline{x}) : \ B_1(\overline{x}), \ldots, B_k(\overline{x})}{C(\overline{x})}$$

where $A(\overline{x}), B_1(\overline{x}), \ldots, B_k(\overline{x})$ and $C(\overline{x})$ are first-order formulas whose free variables are among those of $\overline{x} = (x_1, \ldots, x_n)$.

$A(\overline{x})$ – the *prerequisite*;
$B_1(\overline{x}), \ldots, B_k(\overline{x})$ – the *justifications*;
$C(\overline{x})$ – the *consequent*.

Definition A *default theory* is a pair $T = \langle$W,D$\rangle$, where W is a set of first-order sentences and D is a set of defaults.

A set of theorems derivable from a default theory T is called an *extension of* T.

An extension should include whatever can be inferred from axioms either by ordinary logic or by applying defaults.

If E is an extension of $T = \langle$W,D$\rangle$, then:

(1) E should be deductively closed with respect to standard logic: E=Th(E);

(2) E should contain axioms: W$\subseteq$E;

(3) E should contain a maximal set of conclusions obtainable by applying defaults from D.

Note:

(1)-(3) imply that an extension of a theory $\langle$W,D$\rangle$ is always given by Th(X$\cup$W), where X is a set of conclusions obtainable by applying defaults from D.

Example

W=$\{(1)\ Basket\text{–}Player(Bill)\}$;

D=$\left\{\dfrac{Basket\text{–}Player(x) : Tall(x)}{Tall(x)}, \dfrac{Basket\text{–}Player(x) : Nimble(x)}{Nimble(x)}\right\}$.

E=Th($\{(1), Tall(Bill), Nimble(Bill)\}$).

Let T obtain by adding

(2) $\neg Tall(Bill) \vee \neg Nimble(Bill)$.

E1=Th((1),(2),$Tall(Bill)$);

E2=Th((1),(2),$Nimble(Bill)$).

Definition A default $A(\overline{x}) : B_1(\overline{x}), \ldots, B_k(\overline{x})/C(\overline{x})$ is called *open* iff at least one of $A(\overline{x}), B_1(\overline{x}), \ldots, B_k(\overline{x}), C(\overline{x})$ contains a free variable; otherwise it is called *closed*. A default theory is said to be *open* iff it contains at least one open default; otherwise it is said to be *closed*.

Closed default theories

The following conditions can be naturally imposed on an extension E of a theory $\langle$W,D$\rangle$:

(1) E=Th(E);

(2) W$\subseteq$E;

(3) E should contain a maximal set of conclusions obtainable by applying defaults from D.

Notice:

Any conclusion derivable by applying a closed default is just its consequent.

(3') E should contain the consequent of any applicable default from D.

There still remains the problem of specifying a formal criterion of default's applicability.

$$\frac{A : \ B_1, \ldots, B_k}{C}$$

(3'') If $(A : B_1, \ldots, B_k / C) \in \mathsf{D}$, $A \in \mathsf{E}$ and $\neg B_1 \notin \mathsf{E}, \ldots, \neg B_k \notin \mathsf{E}$, then $C \in \mathsf{E}$.

Closure conditions:

(1) E=Th(E);

(2) W$\subseteq$E;

(3'') If $(A : B_1, \ldots, B_k/C) \in$ D, $A \in$ E and
$\neg B_1 \notin$ E, $\ldots, \neg B_k \notin$ E, then $C \in$ E.

Closure conditions specify what should be in an extension. However, they say nothing about what should be off.

"Definition" E is an extension of a closed default theory iff E is the smallest set satisfying the closure conditions.

"Definition" E is an extension of a closed default theory iff E is a minimal set satisfying the closure conditions.

$T = \langle \{P(a)\}, \{P(a) : Q(a)/Q(a)\} \rangle$

Both

Th($\{P(a), Q(a)\}$) and Th($\{P(a), \neg Q(a)\}$)

satisfy the closure conditions.

Problem:

Assume that E and Γ(E) denote the same extension of $\langle$W,D$\rangle$. How to define Γ(E) by means of E?

Solution:

Let $\Gamma(E)$ be the smallest set satisfying:

(1) $\Gamma(E)=Th(\Gamma(E))$;

(2) $W\subseteq \Gamma(E)$;

(3) If $(A : B_1, \ldots, B_k/C) \in D$, $A \in \Gamma(E)$ and $\neg B_1 \notin E, \ldots, \neg B_k \notin E$, then $C \in \Gamma(E)$.

Definition Let $T = \langle \mathsf{W}, \mathsf{D} \rangle$ be a closed default theory over L_T. For any set of sentences $\mathsf{S} \subseteq L_T$, let $\Gamma(\mathsf{S})$ be the smallest set of sentences from L_T satisfying:

(1) $\Gamma(\mathsf{S}) = \mathsf{Th}(\Gamma(\mathsf{S}))$;

(2) $\mathsf{W} \subseteq \Gamma(\mathsf{S})$;

(3) If $(A : B_1, \ldots, B_k / C) \in \mathsf{D}$, $A \in \Gamma(\mathsf{S})$ and $\neg B_1 \notin \mathsf{S}, \ldots, \neg B_k \notin \mathsf{S}$, then $C \in \Gamma(\mathsf{S})$.

A set of sentences $\mathsf{E} \subseteq L_T$ is an *extension* of T iff $\mathsf{E} = \Gamma(\mathsf{E})$.

Example Reconsider:

$T = \langle \{P(a)\}, \{P(a) : Q(a)/Q(a)\} \rangle$.

T gives rise to two minimal sets satisfying the closure conditions:

E1=Th($\{P(a), Q(a)\}$) and

E2=Th($\{P(a), \neg Q(a)\}$).

Since

Γ(E1)=Th($\{P(a), Q(a)\}$)=E1

Γ(E2)=Th($\{P(a)\}$)$\neq$ E2

only E1 is an extension of T.

Example Let T be a theory with the empty set of axioms and one default

$$\frac{p}{\neg p}.$$

T has no extension. To show it, note that the only candidates are

E1=Th($\{\neg p\}$) and E2=Th($\{\}$).

Since

Γ(E1)=Th($\{\}$)$\neq$E1

Γ(E2)=Th($\{\neg p\}$)$\neq$E2

neither E1 nor E2 is an extension of T.

Theorem If $T = \langle$W,D$\rangle$ is a closed default theory, then a set E of sentences is an extension of T iff E=$\bigcup_{i=0}^{\infty}$ E$_i$, where

E$_0$=W, and for i$\geq$ 0

E$_{i+1}$=Th(E$_i$) $\cup$ $\{C : (A : B_1, \ldots, B_k/C) \in$ D,

where E$_i \vdash A$ and $\neg B_1, \ldots, \neg B_k \notin$ E$\}$.

Corollary A closed default theory $\langle$W,D$\rangle$ has an inconsistent extension iff W is inconsistent.

Theorem If E and F are extensions of a closed default theory and E$\subseteq$F, then E=F.

Open defaults and their instances

An *instance* of an open default is the result of uniformly replacing all free occurrences of its variables by ground terms.

$$\frac{Bird(cousin(Bob)) : \ Flies(cousin(Bob))}{Flies(cousin(Bob))}$$

$$\frac{Bird(x) : \ Flies(x)}{Flies(x)}$$

$$\boxed{T \quad \longrightarrow \quad \text{CLOSED(T)}}$$

Example Let T consist of:

$$Bird(Tweety) \wedge Fish(Nelly)$$

$$\frac{Bird(x) : Has - Wings(x)}{Has - Wings(x)}.$$

CLOSED(T):

$$Bird(Tweety) \wedge Fish(Nelly)$$

$$\frac{Bird(Tweety) : Has - Wings(Tweety)}{Has - Wings(Tweety)}$$

$$\frac{Bird(Nelly) : Has - Wings(Nelly)}{Has - Wings(Nelly)}.$$

Example Let T consist of:

$$Fish(Bob); \quad \forall x.Fish(x) \supset Fish(father(x))$$

$$\frac{Fish(x) : Swims(x)}{Swims(x)}.$$

CLOSED(T):

$$Fish(Bob); \quad \forall x.Fish(x) \supset Fish(father(x))$$

$$\frac{Fish(Bob) : Swims(Bob)}{Swims(Bob)}$$

$$\frac{Fish(father(Bob)) : Swims(father(Bob))}{Swims(father(Bob))}$$

etc.

Example Let T consist of:

$$\exists x.Butterfly(x)$$

$$\frac{Butterfly(x) : Brightly - Coloured(x)}{Brightly - Coloured(x)}.$$

We would like to infer

$$\exists x.Butterfly(x) \wedge Brightly - Coloured(x).$$

CLOSED(T):

$$Butterfly(a)$$

$$\frac{Butterfly(a) : Brightly - Coloured(a)}{Brightly - Coloured(a)}.$$

Example Let T consist of:

$$Birthday(Bill)$$

$$\frac{: \exists x.Friend(x, Bill)}{\exists x.Friend(x, Bill)}$$

$$\frac{Birthday(y) \wedge Friend(x, y) : Gives - Gift(x, y)}{Gives - Gift(x, y)}$$

The first default is replaced by

$$\frac{: \exists x.Friend(x, Bill)}{Friend(b, Bill)}$$

and then we proceed in the usual way.

Definition A *Skolemized form* of a sentence A is any universal sentence obtained from any prenex normal form of A by the process of Skolemization.

Definition A *Skolemized form* of a default

$$\frac{A(\overline{x}) : \ B_1(\overline{x}), \ldots, B_k(\overline{x})}{C(\overline{x})}$$

obtains by replacing $C(\overline{x})$ by any Skolemized form of $\forall C(\overline{x})$ and deleting all quantifiers from the consequent of the resulting default.

Definition A Skolemized form of a default theory T is any theory obtained by replacing defaults and axioms of T by their Skolemized forms.

Definition Let $T = \langle$W,D$\rangle$ be a default theory. By TERMS(T) we denote the set of all ground terms constructable using function constants occurring in W and D.∎

Definition Let T be a default theory. A *closure* of T, written CLOSED(T), is any theory obtained by the following construction:

1. If T is closed, then put CLOSED(T)=T. Otherwise:
2. Replace T by its Skolemized form, $T1$.
3. Let $T2$ be the theory obtained from $T1$ by replacing all its open defaults by their instances over TERMS($T1$). This is the desired closure of T.∎

Definition Let T be a default theory over L_T. E is an extension of T iff E=E'$\cap L_T$ and E' is an extension of CLOSED(T).

Normal default theories

Definition Any default of the form

$$\frac{A(\overline{x}) : B(\overline{x})}{B(\overline{x})}$$

is said to be *normal*. A theory $\langle$W,D$\rangle$ is said to be *normal* iff every default from D is normal.

Theorem Every closed normal default theory has an extension.

Theorem If a closed normal default theory has distinct extensions E and F, then E$\cup$F is inconsistent.

Example Consider:

$$\mathrm{W}=\{\} \quad \mathrm{D}=\left\{\frac{:p}{\neg r}, \frac{:r}{\neg p}\right\}.$$

T has two extensions:

$$\mathrm{E}=\mathrm{Th}(\{\neg r\}) \quad \text{and} \quad \mathrm{F}=\mathrm{Th}(\{\neg p\}).$$

Although E$\cup$F is consistent, $\neg p$ and $\neg r$ should not be regarded as coexisting beliefs.

Theorem (Semi-Monotonicity) Let D and D' be sets of closed normal defaults such that D'$\subseteq$D. Let E' be an extension of $T'=\langle$W,D'$\rangle$ and suppose that $T=\langle$W,D$\rangle$. Then T has an extension E such that E'$\subseteq$E.

Algorithm

T=⟨W,D⟩ – closed default theory with a finite set of defaults.

PERM – the set of all permutations of elements from D.

If PERM={}, i.e. D={}, or if W is inconsistent, then return Th(W) as the only extension of T **else**

while PERM≠ {} **do**

begin

(1) Take any permutation perm=($d_1, \ldots, d_n$) from PERM;
PERM:=PERM–{perm};

(2) [**Initialization]**
BEL:=W; JUST:={};

(3) **[Application of defaults+Consistency test]**
repeat
for i:=1 **to** n **do** $\{$ d$_i = A_i : B_i/C_i.\}$
If BEL$\vdash A_i$ and BEL$\nvdash \neg B_i$, **then**
begin
BEL:=BEL$\cup\{C_i\}$;
JUST:=JUST$\cup\{B_i\}$;
If there is $A \in$JUST such that BEL$\vdash \neg A$, **then**
goto (5)
end
until the set BEL is *stable* i.e. it has not changed during the "for" loop;

(4) **return** Th(BEL) as an extension;

(5)

end while

Representational Issues

To what extend normal defaults are sufficient for the purpose of practical applications?

1. Bill is a high school dropout.
2. Typically, high school dropouts are adults.
3. Typically, adults are employed.

W=$\{Dropout(Bill)\}$

D=$\left\{ \dfrac{Dropout(x) : Adult(x)}{Adult(x)}, \quad \dfrac{Adult(x) : Employed(x)}{Employed(x)} \right\}$.

This theory forces us to believe that Bill is employed.

Solution:

$$\frac{Adult(x) : Employed(x) \wedge \neg Dropout(x)}{Employed(x)}$$

■

1. Bill is a 21-year-old.
2. All 21-year-olds are adults.
3. Typically, adults are married.

$21 - Year - Old(Bill)$

$\forall x.21 - Year - Old(x) \supset Adult(x)$

$$\frac{Adult(x) : Married(x)}{Married(x)}.$$

This theory forces the conclusion that Bill is married.

$$\frac{Adult(x) : Married(x) \wedge \neg 21 - Year - Old(x)}{Married(x)}$$

The general situation: We are given a rule which in isolation is naturally represented by a normal default

$$\frac{A(\overline{x}) : B(\overline{x})}{B(\overline{x})}.$$

When the rule is embedded in the context, there are some exceptional circumstances, say $E(\overline{x})$, which make the rule inapplicable. The simplest solution is to replace the normal default by

$$\frac{A(\overline{x}) : B(\overline{x}) \wedge \neg E(\overline{x})}{B(\overline{x})}.$$

Definition Any default of the form

$$\frac{A(\overline{x}) : B(\overline{x}) \wedge C(\overline{x})}{B(\overline{x})}$$

is called *semi-normal*. A default theory is *semi-normal* iff all of its defaults are semi-normal. Note that a normal default $A : B/B$ can be identified with the semi-normal default $A : B \wedge True/B$.

Semi-normal default theories may lack extensions:

Let $T = \langle\{\}, \mathrm{D}\rangle$, where D consists of

$$\mathrm{d1}=\frac{: p \wedge \neg q}{\neg q}, \qquad \mathrm{d2}=\frac{: q \wedge \neg r}{\neg r}, \qquad \mathrm{d3}=\frac{: r \wedge \neg p}{\neg p}.$$

In general, semi-normal theories lack the property of semi-monotonicity:

Let $T = \langle \{\}, : p \wedge q / q \rangle$. T has one extension:

E=Th($\{q\}$).

Adding : $\neg p / \neg p$ to T, we arrive in a theory with one extension given by

F=Th($\{\neg p\}$).

(1) Are semi-normal theories sufficient to cover all naturally occurring practical applications?

(2) To what extend can the effects obtainable by using semi-normal defaults be achieved in the framework of normal representation?

Translation schema

(1) Replace any default of the form

$$\text{d}=\frac{A(\overline{x}) : B(\overline{x})}{C(\overline{x})}$$

by the semi-normal default

$$\text{d1}=\frac{A(\overline{x}) : B(\overline{x}) \wedge C(\overline{x})}{C(\overline{x})}.$$

This step seems to be intuitively sound what leads to the following observation: **Any common-sense setting which can be correctly modelled by using single-justification defaults may be expressed by means of semi-normal defaults.**

Important: A default and its translation need not be equivalent. Consider d=: $p/\neg p$ and its translation d1=: $p \wedge \neg p/\neg p$. While the theory $\langle\{\},\{\text{d}\}\rangle$ lacks an extension, the theory $\langle\{\},\{\text{d1}\}\rangle$ has an extension given by Th({}).

(2) Replace the semi-normal default

$$\text{d1}=\frac{A(\overline{x}) : B(\overline{x}) \wedge C(\overline{x})}{C(\overline{x})}$$

by the normal default

$$\text{d1}=\frac{A(\overline{x}) : B(\overline{x}) \wedge C(\overline{x})}{B(\overline{x}) \wedge C(\overline{x})}.$$

This works if we accept: "Typically, if $A(\overline{x})$ and $C(\overline{x})$, then $B(\overline{x})$.

$$\frac{Adult(x) : Married(x) \wedge \neg 21 - Year - Old(x)}{Married(x)}$$

$$\frac{Adult(x) : Married(x) \wedge \neg 21 - Year - Old(x)}{Married(x) \wedge \neg 21 - Year - Old(x)}$$

Here is an example where the schema does not work.

$$\frac{Has - Motive(x) : Guilty(x)}{Suspect(x)}$$

$$\frac{Has - Motive(x) : Guilty(x) \wedge Suspect(x)}{Suspect(x)}$$

$$\frac{Has - Motive(x) : Guilty(x) \wedge Suspect(x)}{Guilty(x) \wedge Suspect(x)}$$

Advanced topics on default logic, (not disscussed here)

- Semantic characterization of extensions (Łukaszewicz 1985, Etherington 1987).
- Proof theory for DL (Reiter 1980).
- Connections between DL and AEL (Konolige 1988, Marek & Truszczynski 1989).
- Alternative formulations of default logic:
  - Łukaszewicz 1984.
  - Brewka 1991.

Conclusions

Advantages of DL:

Conceptual clarity.

Weaknesses of DL:

(1) Lack of natural semantics.

(2) Some theories lack extensions.

(3) DL cannot deal with reasoning by cases. For instance, given

W=$\{bird \vee rocket\}$; D=$\left\{\frac{bird : flies}{flies}, \frac{rocket : flies}{flies}\right\}$

we intuitively feel that $flies$ should be derivable. Unfortunately, it is not.