Spectra of formulae with restrictions

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Spectrum (Scholz '52)

Let ϕ be a formula (usually FO).
Then the spectrum of ϕ (denoted $\mathrm{spec}(\phi)$) is the set of $N \in \mathbb{N}$ such that ϕ has a model of size N.

Examples of spectra

Example

ϕ = conjunction of axioms of linear spaces over $\mathbb{Z}_2$

$\mathrm{spec}(\phi)$ = powers of 2

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The main problem: Characterize subsets of $\mathbb{N}$ which are $\mathrm{spec}(\phi)$ for some ϕ.

Asser's problem

$$\mathrm{spec}(\phi \wedge \psi) = \mathrm{spec}(\phi) \cap \mathrm{spec}(\psi)$$

$$\mathrm{spec}(\phi \vee \psi) = \mathrm{spec}(\phi) \cup \mathrm{spec}(\psi)$$

(if ϕ and ψ are over disjoint signatures)

Thus, spectra are closed under (finite) union and intersection.

Asser's problem ('55):
What about negation?

Idea (Fagin '74, Jones & Selman '74):
Characterize spectra using **complexity theory**.

Descriptive complexity:
find relationships between computational (complexity theoretic) and descriptive (logical) characterizations of objects.

Integers and languages

We identify subsets of $\mathbb{N}$ with subsets of $\{0,1\}^*$:

$$A \subseteq \mathbb{N}$$

$$\Leftrightarrow$$

$$\{\mathbf{bin}(N) : N \in A\}$$

... and run computations on these binary encodings.

Notation

N - the number, n - the length of its encoding ($N = \Theta(2^n)$)

Notation

- **NTIME**$(f(n))$ – the class of decision problems which can be solved non-deterministically in time $O(f(n))$
- e.g. **NTIME**(2^n) - pseudolinear time
  NTIME(2^{2n}) - pseudoquadratic time
- **NE** $= \bigcup_k$ **NTIME**(2^{nk})
  (non-deterministic pseudopolynomial time)
- note **NE** $\neq$ **NEXPTIME** $= \bigcup_k$ **NTIME**$(2^{n^k}))$
- **NTISP**$(f(n), g(n))$ – the class of decision problems which can be solved non-deterministically in time $O(f(n))$ and space $O(g(n))$

Fagin's theorem

Theorem (Fagin '74, Jones & Selman '74)

$$\mathsf{SPEC} = \mathsf{NE}$$

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$$\mathbf{SPEC} = \mathbf{NE}$$

For a formula ϕ using at most k variables and k-ary relations, $\mathrm{spec}(\phi) \in \mathbf{NTIME}(n2^{nk})$ (guess the model and just check whether the formula is satisfied)
On the other hand, let $A \in \mathbf{NTISP}(2^{nk}, 2^{nl})$

- ϕ defines an order on $\{1, \ldots, N\}$
- we encode the computation of the machine on a $N^l \times N^k$ grid $G(x, y)$
- we encode G with $R(x_1, \ldots, x_l, y_1, \ldots, y_k)$
- in ϕ, we check whether R is correct (using $l + k + 1$ variables)

Corollaries and generalizations

- Spectra are closed under complement iff $\mathbf{NE} = \mathbf{coNE}$ (equivalently, $\mathbf{NP} \cap \mathbf{TALLY} = \mathbf{coNP} \cap \mathbf{TALLY}$)

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- **Generalized spectra** (Fagin)
  Classes of structures, not numbers – we know some relations $R_1, \ldots, R_k$ and want to know if $R_{k+1}, \ldots$ exist such that ϕ holds
  Corollary: a class of structures is definable in $\exists SO$ iff it is decidable in **NP** (big open problem: is there a logic which captures **P** in the same way?)

Corollaries and generalizations, part 2

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  $\Psi(\phi) = \{(N_1, \ldots, N_k) : \phi$ has a model where N_i elements satisfy $P_i\}$

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- *Fifty years of the spectrum problem: survey and new results.* Durand, Jones, Makowsky, More, 2009
- Okay, we know a lot about spectra – but what if we put **restrictions?**

Restricted Variables

Let $\mathbf{FO}^k$ be the class of formulae using k variables.

$$\exists x \exists y (\dots \exists x \dots) \in \mathbf{FO}^2$$

(we consider only relational signatures)

Let $\mathbf{SPEC}(\mathbf{FO}^k)$ be the class of spectra of $\mathbf{FO}^k$ formulae.
Does the hierarchy collapse?

Variable hierarchy

Theorem (K, Tony Tan)

$$\begin{aligned} \mathrm{SPEC}(\mathrm{FO}^k) &\subseteq \mathrm{NTIME}(n2^{kn}) \subsetneq \\ &\subsetneq \mathrm{NTIME}(2^{(k+1/2)n}) \subseteq \mathrm{SPEC}(\mathrm{FO}^{2k+2}) \end{aligned}$$

(listed as an open problem in the survey)

Variable hierarchy: two variables with counting

Between $\mathbf{FO}^2$ and $\mathbf{FO}^3$: $\mathbf{C}^2$ (two-variable logic with counting)

- we can use 2 variables
- we can also say *there are at least k elements such that ...*

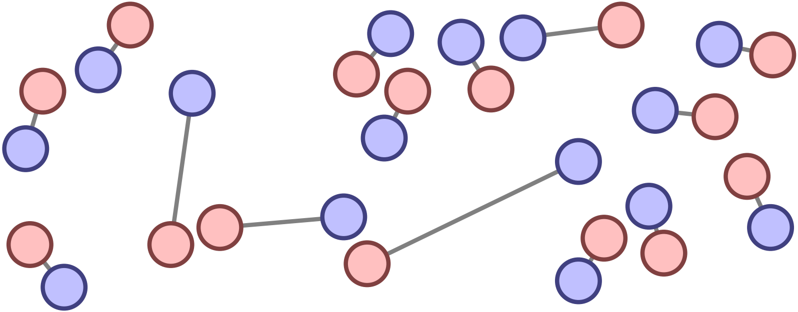
$$\forall x \exists_{=2} y R(x, y)$$

This is a logic with good properties:

- Decidable (Grädel, Otto, Rosen '97)
- Related to modal logic

Two variables with counting – examples

- ϕ_1: *Each man is married to exactly one woman, each woman is married to exactly one man*

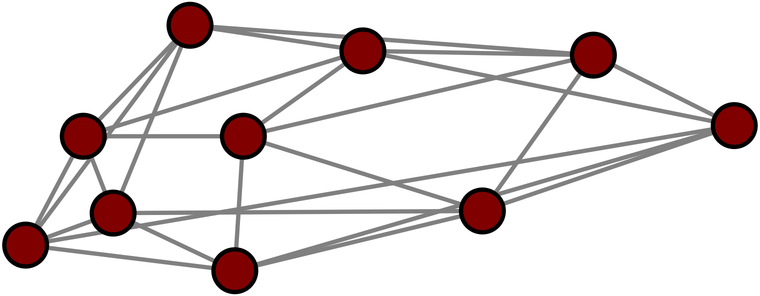


$$\Psi(\phi_1) = \{(n_1, n_2) : n_1 = n_2\}$$

(a 1-regular bipartite graph)

Two variables with counting – examples

- ϕ_2: *Each mafia member has contact to exactly five other mafia members*

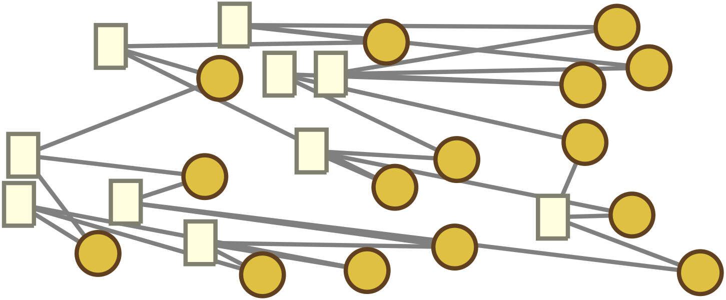


$$\mathrm{spec}(\phi_2) = \{n : 2|n, n \neq 2, n \neq 4\}$$

(a 5-regular graph)

Two variables with counting – examples

- ϕ_3: *Each paper has three authors, each author has written two papers*

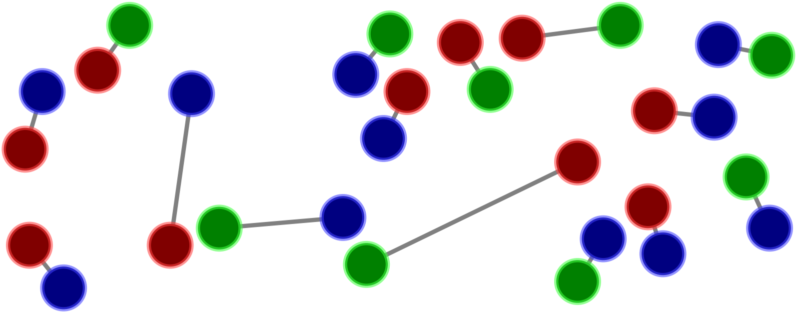


$$\Psi(\phi_4) = \{(n_1, n_2) : 3n_1 = 2n_2\}$$

(a 2,3-regular bipartite graph)

Two variables with counting – examples

- ϕ_4: *There are three teams, each competitor plays against a competitor from another team*



$$\Psi(\phi_3) = \left\{ (n_1, n_2, n_3) : \begin{array}{l} n_1 \leq n_2 + n_3, n_2 \leq n_3 + n_1, \\ n_3 \leq n_1 + n_2, 2 | n_1 + n_2 + n_3 \end{array} \right\}$$

C^2: the main result

Theorem (K, Tony Tan)

Let ϕ be a formula of two-variable logic with counting. Then $\Psi(\phi)$ is definable in Presburger arithmetic.

Sets definable in Presburger arithmetic are exactly the **semilinear sets**.

Semilinear set = a union of finitely many **linear sets**

Linear set = a set of form $\{b + n_i v_i : n_1, \ldots, n_k \in \mathbb{N}\}$ for some $b, v_1, \ldots, v_k$

$\mathbf{C}^2$: corollaries

Note that for each semilinear set S it is easy to construct a $\mathbf{C}^2$ formula ϕ such that $S = \Psi(\phi)$.

Corollary

A set of positive integers is a spectrum of a $\mathbf{C}^2$ formula iff it is eventually periodic.

Corollary

$\mathbf{C}^2$ *spectra (and images) are closed under complement.*

Proof: simplify the universe

Let ϕ be a $\mathbf{C}^2$ formula.
We can assume that:

- ϕ is over a signature including only unary relations $\mathcal{P} = \{P_1, \ldots, P_d\}$ and binary relations $\mathcal{R} = \{R_1, \ldots, R_l\}$
- for each two elements x, y, either $x = y$ or $R_i(x, y)$ for exactly one relation R_i
- For each relation $R \in \mathcal{R}$ there is a reverse relation $\overleftarrow{R} \in \mathcal{R}$, such that $R_i(x, y)$ iff $\overleftarrow{R_i}(y, x)$.

Proof: use modal logic

We transform ϕ into a formula of QMLC (quantified modal logic with counting).

$$\textbf{MLC}: \psi ::= \neg\psi \mid P \mid \psi_1 \wedge \psi_2 \mid \Diamond^k_R \psi$$

$a \models \Diamond^k_R \psi$ iff there are at least k elements b such that $R(a, b)$ and $b \models \psi$

$$\textbf{QMLC}: \phi ::= \neg\phi | \phi_1 \wedge \phi_2 | \exists^k \psi$$

where $\exists^k \psi$ (where $\psi \in$ MLC) means that there are at least k elements a such that $a \models \psi$

Proof: C, D-regular bipartite graphs

Is there a complete bipartite graph such that:

- there are $M_1 + \ldots + M_m$ vertices on the left side
- there are $N_1 + \ldots + N_n$ vertices on the right side
- each edge has one of l colors
- each of the M_i vertices has $C_{i,j}$ edges of color j
- each of the N_i vertices has $D_{i,j}$ edges of color j?
- $C_{i,j}$ and $D_{i,j}$ can be given as an *exact* number or *at least* some number: $\mathbb{B} = \{=0, =1, =2, \ldots, =k, \geq 0, \geq 1, \ldots, \geq k\}$

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Theorem

There is a Presburger formula $\Psi_{C,D}(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ such that $\Psi_{C,D}(M_1, \ldots, M_m, N_1, \ldots, N_n)$ holds iff such a graph exists

Proof: types and functions

Each element a of the universe has a *type* (the set of MLC subformulae of ϕ which are satisfied in a). Let $\mathcal{T}$ be the set of all types.

Let $X_{T,f}$ be a variable (intuitively, the number of elements of type T whose number of edges to other types is given by a function $f : \mathcal{R} \times \mathcal{T} \to \mathbb{B}$; we consider only functions consistent with the semantics of T).

For each two types T_1, T_2 we use the previous Theorem to generate Presburger formulas to verify whether $X_{T_1,f}$ and $X_{T_2,f}$ are consistent. We also need another theorem for the case where $T_1 = T_2$.

Summary: variable hierarchy and spectra

- $\mathbf{FO}^1$: empty and $\{n : n \geq k\}$
- $\mathbf{C}^1, \mathbf{FO}^2$: finite and cofinite sets
- $\mathbf{C}^2$: semilinear sets
- $\mathbf{FO}^3 \supseteq \mathbf{NTIME}(2^n)$ (pseudolinear), $\subseteq \mathbf{NTIME}(n2^{3n})$
- $\mathbf{FO}_k \subsetneq \mathbf{FO}_{2k+2}$

Graph structure

$\mathbb{A}$ - structure
Gaifman graph of $\mathbb{A}$ - graph whose vertices are the elements of universe of A, edges are vertices which are related
Many logical and algorithmic properties of graphs are easier when the graph is simple:

- bounded degree
- planar
- bounded tree width
- forbidden minor

Graph structure: trees

- For signatures including only unary relations and one unary function, spectra can only be semilinear sets (Durand, Fagin, Loescher '97)

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- This is because the model has to be of form of unconnected cycles with trees (with two unary functions spectra can be **NEXPTIME**-complete)

Graph structure: bounded tree width

- If all models of a CMSO (MSO with modulo counting) formula ϕ have bounded tree width, then (many-sorted) $\mathrm{spec}(\phi)$ is semilinear (Fischer, Makowsky 2004)
- This holds for images, too

Bounded degree spectra

A **degree d spectrum** is a set $S \subseteq \mathbb{N}$ such that $S = \mathrm{spec}(\phi)$ for some formula ϕ such that all models of ϕ are of degree at most d.

Equivalently: a set $S \subseteq \mathbb{N}$ such that for some formula ϕ, $n \in S$ iff ϕ has a model with degree at most d.

By $\mathbf{BDSpec}_d$ we denote the set of all degree d spectra.

Our result

For $d \geq 3$,

Theorem (Anuj Dawar, K)

$$\mathbf{NTIME}(2^n) \subseteq \mathbf{BDSpec}_d \subseteq \mathbf{NTIME}(n^2 2^n)$$

- for the first inclusion, we only require $d = 3$, unary relations, and a single symmetric binary relation.

Technique: $\mathbf{NTIME}(2^n) \subseteq \mathbf{BDSpec}_d$

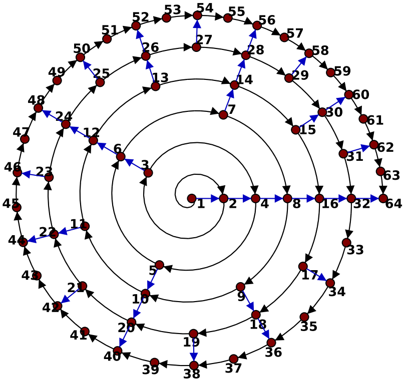
How to simulate Turing machines with bounded degree spectra?
We use partial injective function symbols (PIFs)

$$k \text{ PIFs} \rightarrow \text{degree } 2k$$

Integers

We can axiomatize the following structure:

$$A = \{1, \ldots, N\}$$
$$f_A(x) = x + 1$$
$$g_A(x) = 2x$$



Integer functions

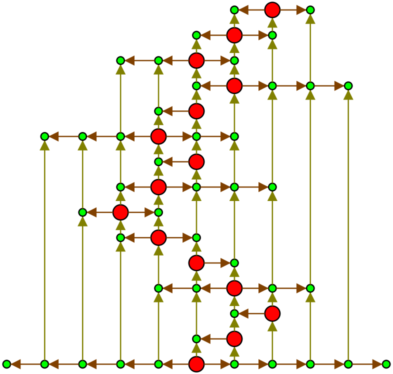
We can add the following functions (where C is a constant in the structure):

$$h(x) = x + C \quad (1)$$
$$j(x) = x \times C \quad (2)$$
$$k(x) = 2^x \quad (3)$$
$$l(x) = \lfloor N/2^x \rfloor \quad (4)$$

... and use l to find the binary representation of N, and h to construct a Turing machine working on it in space C and time N/C

Effective simulation of a Turing machine

With more sophisticated techniques this can be optimized to time $O(N)$:



Technique: $\mathbf{BDSpec}_d \subseteq \mathbf{NTIME}(n^2 2^n)$

To check whether $N \in \mathrm{spec}(\phi)$:

- We guess the structure $\mathfrak{A}$ of size N and degree d
- We use Hanf's locality theorem to effectively verify whether $\mathfrak{A}$ satisfies ϕ

Theorem (Hanf '65)

Let ϕ be a FO formula. Then there exist numbers r and M such that, for each graph $G = (V, E)$, $G \models \phi$ depends only on the number of r-neighborhoods of each type, up to the threshold of M.

Two definitions of planar spectra

There are two non-equivalent definitions of planar spectra:

- A *planar* spectrum of formula ϕ, $\mathrm{pspec}(\phi)$, is the set of cardinalities of all models of ϕ whose Gaifman graph is planar
- **PSpec** is a set of all $S \subseteq \mathbb{N}$ such that $S = \mathrm{pspec}(\phi)$ for some ϕ
- **FPSpec** if a set of all $S \subseteq \mathbb{N}$ such that $S = \mathrm{spec}(\phi)$ for some ϕ such that all models of ϕ are planar

Weak planar spectrum: lower bounds

Theorem (Anuj Dawar, K)

$$\mathbf{NTIME}^S(2^n/n) \subseteq \mathbf{PSpec}$$

- our model simulating a Turing machine is planar as long as the machine has only one tape (superscript S)
- we can make sure that all members of the universe are a part of this simulation (as long as we only consider planar models!)
- we are unable to read the size of the universe (that required non-planarity), but we can calculate it with a logarithmic overhead

Weak planar spectrum: queue machines

Let M be a non-deterministic non-contracting queue machine
M accepts $n \in \mathbb{N}$ iff M has a computation which writes n symbols to the queue
QTL $\subseteq P(\mathbb{N})$ is the class of sets of integers accepted by some queue machine of this type

Example

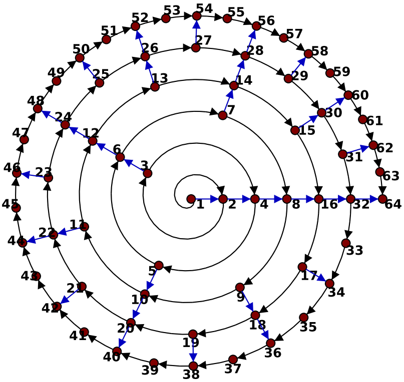
Initial queue contents: A, transitions:

$$\begin{aligned} A &\rightarrow Aa \\ a &\rightarrow aa \\ A &\rightarrow \text{accept} \end{aligned}$$

accepts powers of two minus one

We write all elements in the queue in a line

Black arrows: next in the queue Blue arrows: connect the reading head to the writing head



Weak planar spectrum: queue machines

Theorem (Anuj Dawar, K)

$$\mathbf{NTIME}(\sqrt{2^n}) \subseteq \mathbf{NTS}(2^n) \subseteq \mathbf{QTL} \subseteq \mathbf{PSpec}$$

$\mathbf{NTS}(2^n)$ = the product of time and space is $O(2^n)$

Weak planar spectrum: upper bounds

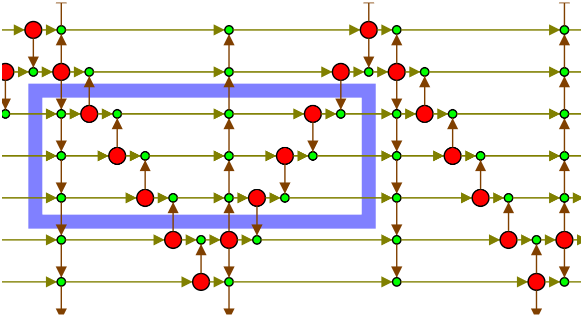
Our planar models so far have bounded degree ($\mathbf{PBDSpec}_d$).

$$\mathbf{PBDSpec}_d \subseteq \mathbf{NTIME}(2^n n^2)$$

Without assuming bounded degree, on a non-deterministic RAM machine, in linear time we can guess the model M, verify whether M is a planar graph and whether M satisfies ϕ (Frick, Grohe 2001). Together with the time hierarchy theorem, this solves an open problem from the *Fifty years of the spectrum problem*.

Forcing planarity

We cannot use our simulation of Turing machines when we require all models of ϕ to be planar:



Forcing planarity

We cannot use our simulation of queue machines either:

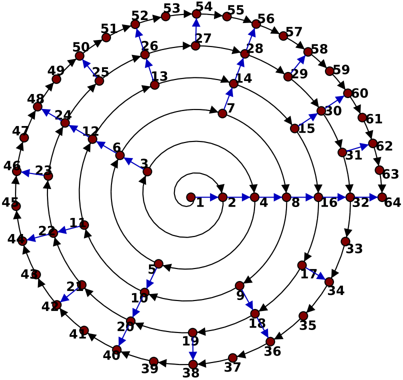
$$w \to^* w$$

We can write this computation on a torus

We could simulate an "extending" queue machine:

$$w \to^* u \text{ implies } |u| > |w|$$

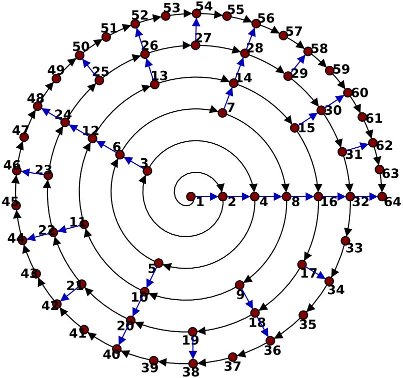
Forcing planarity



However, such queue machines typically blow up the queue exponentially in each iteration, and thus we get only a logarithmic number of iterations.

Forcing planarity

By using the $\log_d(N)$ outermost layers of this spiral, we can calculate the d-ary representation of N, and run a Turing machine in space $\log_d(N)$ and time $N^{1-\log_d 2}$



Forcing planarity

Theorem (Anuj Dawar, K)

$$\mathbf{NTISP}((2-\epsilon)^n, n) \subseteq \mathbf{FPSpec}$$

We can simulate a Turing machine which recognizes the binary representation of N in space $\log N$ and time $N^{1-\epsilon}$.

Example

the set of primes (via the trivial algorithm)

Open problems

- Finer hierarchy than $\mathbf{SPEC}(\mathbf{FO}^k) \subsetneq \mathbf{SPEC}(\mathbf{FO}^(2k+2))$
- Between $\mathbf{C}_2$ and $\mathbf{FO}_3$: $\mathbf{C}_2(<)$
  We know that $\Psi(\phi)$ for $\phi \in FO_2C(<)$ include reachability sets of Petri nets (so no longer semilinear, but still decidable – Kosaraju '82)
- Arity hierarchy
- Can we reduce the gaps between the lower and upper bounds?
- Can we use more memory in the forced planar case?
- What about other classes of graphs with an excluded minor?

Summary

- Variable hierarchy: $\mathbf{SPEC}(\mathbf{FO}^k) \subsetneq \mathbf{SPEC}(\mathbf{FO}^{2k+2})$
- Two variables and counting: semilinear sets
- Bounded treewidth: semilinear sets
- Bounded degree: $\mathbf{NTIME}(2^n) \subseteq \mathbf{BDSpec}_d \subseteq \mathbf{NTIME}(n^2 2^n)$
- Weak planarity: $\mathbf{NTIME}_S 2^n/n \subseteq \mathbf{PSpec}$, $\mathbf{PSpec} \subseteq \mathbf{NTIME}(c^n)$
- Forced planarity: $\mathbf{NTISP}((2-\epsilon)^n, n) \subseteq \mathbf{FPSpec}$