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Half-positional Determinacy of Infinite Games

PhD Thesis

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Author's declaration:
aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Supervisor's declaration:
the dissertation is ready to be reviewed

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Abstract

We study infinite games where one of the players always has a positional (memory-less) winning strategy, while the other player may use a history-dependent strategy. We investigate winning conditions which guarantee such a property for all arenas, or all finite arenas. We establish some closure properties of such conditions, which give rise to the XPS class of half-positional winning conditions, and discover some common reasons behind several known and new positional determinacy results. We show that this property of half-positional determinacy is decidable in single exponential time for a given prefix independent ω-regular winning condition. We exhibit several new classes of half-positional winning conditions: the class of concave conditions (for finite arenas), the classes of monotonic conditions and geometrical conditions (for all arenas).

Keywords

automata, infinite games, omega-regular languages, positional strategies, winning conditions

AMS Classification:

68Qxx Theory of computing
68Q45 Formal languages and automata
68Q60 Specification and verification
91Axx Game theory
91A05 2-person games
91A43 Games involving graphs

Streszczenie

Badamy gry nieskończone, w których jeden z graczy ma zawsze pozycyjną (bezpamięciową) strategię wygrywającą, podczas gdy drugi graczy może używać strategii zależnej od historii. Badamy warunki zwycięstwa gwarantujące taką własność dla wszystkich aren, oraz dla wszystkich skończonych aren. Pokazujemy warunki domknięcia tej klasy warunków zwycięstwa, prowadzące do klasy XPS warunków półpozycyjnych, a także znajdujemy wspólne powody dla kilku znanych i nowych wyników dotyczących pozycyjnej determinacji. Pokazujemy, że własność półpozycyjnej determinacji danego ω-regularnego warunku zwycięstwa jest rozstrzygalna w czasie wykładniczym. Pokazujemy kilka nowych klas warunków półpozycyjnych: warunki wklęsłe (dla aren skończonych), monotoniczne i geometryczne (dla aren o dowolnej mocy).

Słowa kluczowe

automaty, gry nieskończone, języki omega-regularne, strategie pozycyjne, warunki zwycięstwa

Klasyfikacja według ACM:

68Qxx Teoria obliczeń
68Q45 Języki formalne i automaty
68Q60 Specyfikacja i weryfikacja
91Axx Teoria gier
91A05 Gry dla 2 graczy
91A43 Gry związane z grafami

Contents

Chapter 1

Introduction

The theory of infinite games is relevant for computer science because of its potential application to verification of interactive systems. In this approach, the system and environment are modeled as players in an infinite game played on a graph (called *arena*) whose vertices represent possible system states. The players (conventionally called Eve and Adam) decide which edge (state transition, or *move*) to choose; each edge has a specific *color*. The desired system's behavior is expressed as a winning condition of the game — the winner depends on the sequence of colors which appear during an infinite play. If a winning strategy exists in this game, the system which implements it will behave as expected. *Positional* strategies (i.e. depending only on the position, not on the history of play — also called *memoryless*) are of special interest here, because of their good algorithmic properties which can lead to an efficient implementation. Among the most often used winning conditions are the parity conditions, which admit positional determinacy for both players ([Mos91], [EJ91], [McN93]).

Infinite games are also strongly linked to automata theory. Parity condition is a very important notion in both fields — infinite games and automata on infinite structures. Winning conditions in games can often be effectively expressed as ω-regular languages. This allows results from one field to be used in another. For example, positional determinacy of parity games is used in the modern proofs of Rabin's complementation theorem for finite automata on infinite trees with parity acceptance condition.

However, not always it is possible to express the desired behavior as a parity condition. An interesting question is, what properties are enough for the winning condition to be positionally determined, i.e. admit positional winning strategies independently on the arena on which the game is played. Recently some interesting characterizations of such positionally determined winning conditions have been found ([CN06], [GZ05]). Another interesting characterization of finitely positional conditions can be found in [GZ04]. For a survey of recent results on positional determinacy see [Gra04].

Our work attempts to obtain similar characterizations and find interesting properties (e.g. closure properties) of half-positionally determined winning conditions, i.e. ones such that all games using such a winning condition are positionally determined for one of the players (us, say), but the other player (environment) can have an arbitrary strategy. We give uniform arguments to prove several known and several new half-positional determinacy results. As we will see, some results on positional determinacy have natural generalizations to half-positional determinacy, but some do not. This makes the theory of half-positional conditions harder than the theory of positional conditions. We also exhibit some large classes of half-positionally determined winning conditions.

1.1 Overview

Chapter 2 In this chapter we begin with some examples of infinite games, with positional and non-positional winning strategies. Then we proceed to introduce the basic definitions and notions we will be using throughout the thesis, like winning conditions, arenas, games, strategies, and positional strategies. We introduce basic determinacy types, like positional and half-positional determinacy. We define a half-positional winning condition as one which admits positional strategy for Eve no matter what arena is this winning condition used on, and discuss how this class of half-positional winning conditions changes for various classes of arenas that appear in literature (arenas can have labels on edges, on positions, or on only a subset of positions).

Chapter 3 In this chapter we present tools which can be used to prove (half-) positional determinacy of many winning conditions in an uniform way. We start with some basic properties of positional strategies. Although these properties are most interesting for positional strategies, we present the proof in a more abstract way which also encompasses arbitrary strategies. These properties are quite well known by the researchers in this field, and are the reason why we concentrate on prefix independent winning conditions (as they need not work for prefix dependent winning conditions). Then, we use these properties to show Lemma 3.5, which we will use to show half-positional determinacy of many winning conditions in the sequel. Again, Lemma 3.5 is presented in an abstract way, thus it can be used to prove both half-positional and positional determinacy. We use Lemma 3.5 to prove that if W is (half-) positional, then so is $W \cup WB_S$ (Theorem 3.7); the latter winning condition says that Eve wins if she wins W or if colors from S appear infinitely many times. (WB_S is a Büchi condition: Eve wins iff colors from S appear infinitely many times.) Theorem 3.7 leads to an alternative proof that the parity conditions are positionally determined. We conclude Chapter 3 by quoting and generalizing some previously known characterizations [CN06, GZ05] of positional and finitely positional winning conditions.

Chapter 4 We present a simple combinatorial property, *concaveness*, which guarantees *finite* (but not infinite) half-positional determinacy. Namely, a wining condition is concave iff whenever Adam wins if the sequence of colors during an infinite play is w_1 or w_2, he also wins for all shuffles of w_1 and w_2. This result is strongly related to its positional counterpart from [GZ04] about fairly mixing payoff mappings. We also note show relations between our theorem and the result from [MT02] about positive winning conditions and persistent strategies.

Chapter 5 Here, we generalize the mean payoff game to many dimensions. In our game, we let our set of colors be $C = [0, 1]^d$; our winning conditions are defined in terms of the sequence whose n-th term is the average of the first n colors visited during our infinite play. We say that Eve wins $WF(A)$ iff each cluster point of this sequence is in $A \subseteq C$, and she wins $WF'(A)$ iff at least one cluster point is in A. We investigate for which A's the winning conditions $WF(A)$ and $WF'(A)$ are concave, convex, weakly concave and weakly convex (as defined in Chapter 4), and for which A's they are (finitely) half-positional or positional. Namely, $WF'(A)$ is finitely half-positional for A which is a complement of a (geometrically) convex subset of C, and, for infinite arenas, $WF(A)$ is half-positional for $A = [0, 1/2)$ (Theorem 5.7).

Chapter 6 In this Chapter we explore the links between games and automata theory. We define a *monotonic automaton* as one whose set of states is $Q = \{0, \ldots, n\}$, and whose transition function is monotonic. In Theorem 6.6 we show that a winning condition WM_A defined in terms of a monotonic automaton A is half-positional. Further results of this chapter deal with ω-regular winning conditions, i.e., ones defined in terms of a DFA with parity acceptance condition. In Theorem 6.9 we show that if such a ω-regular winning condition is not half-positional, then this fact is witnessed by a very simple witness arena, namely one in which Eve has a choice in only one position, and she has a choice between only two moves there. Then we use this characterization in Theorem 6.10 to present an algorithm which decides half-positional determinacy for an ω-regular winning condition; this algorithm runs in single exponential time. We conclude this chapter with PTIME decidability of concavity of ω-regular winning conditions.

Chapter 7 In Chapter 7 we present one of the questions which motivated our research: is a finite (countable) union of half-positional winning conditions also half-positional? In Theorem 7.2 we show that this fails for uncountable unions: We show an example of an uncountable family of half-positional winning conditions (in fact, even positional, and very simple — Büchi and co-Büchi) whose union is not half-positional. The conjecture is still open for finite and countable unions, but we have some partial results. We define *suspendable winning strategies*, which, intuitively, allow the player using them to sometimes suspend using them, and return to them later; and the player will still win if he is doing that correctly. We define *positional/suspendable winning conditions* as ones which admit positional winning strategies for Eve and suspendable winning strategies for Adam. We show that some of the previously mentioned half-positional winning conditions are in fact positional/suspendable, namely, co-Büchi conditions, monotonic conditions, and some of the geometrical conditions. In Theorem 7.10 we have shown that a union of countably many positional/suspendable winning conditions is also positional/suspendable. We proceed with defining yet another class of winning conditions, XPS (*extended positional/suspendable winning conditions*), which contains all positional/suspendable and parity winning conditions, and is closed under finite union, and intersection with co-Büchi conditions. This class contains most (or all?) of half-positional winning conditions mentioned in this thesis, and in Theorem 7.12 we have shown that all XPS winning conditions are half-positional. We conclude this chapter with Theorem 7.13, which shows that each winning condition that can be presented as a finite union of monotonic and concave winning conditions is half-positional.

Chapter 8 Here we investigate games where we cannot use a positional (memoryless) strategy, and we require another, weaker property for Eve's strategy instead. We investigate how some of our results from the previous chapters can be extended to these weaker kinds of strategies. There are two kinds of such strategies. One possibility is to use the smallest amount of memory possible. We present a definition of a strategy with memory, and show that it is possible to calculate the smallest (chromatic) memory size for ω-regular winning conditions (Theorems 8.13 and 8.14). The second possibility is *persistent strategies*, as introduced in [MT02]. Just like a positional strategy, a persistent strategy always uses the same move in each position; however, contrary to a positional strategy, this move is decided not before game, but when the play visits this position for the first time. We show some examples (8.19, 8.20) of winning conditions which are half-persistent, but not half-positional for some classes of arenas, and we show Theorem 8.24, which is a generalization of Theorem 3.7 (about taking an union with a Büchi condition) for half-persistent strategies. This chapter is a work in progress and has more open paths than the previous chapters.

Chapter 9 We recollect all the open problems and areas of further research which have arisen during the work on this dissertation.

Finally, on page 87 there is a notation index which lists all the notation commonly used thorough the thesis, together with their meanings and page numbers where they have been defined. Also at the end of the thesis is the usual index and bibliography.

1.2 Acknowledgments

First of all, I am grateful to my supervisor Damian Niwiński for showing me the beautiful field of infinite games, introducing me to beautiful problems, sending me to meetings, schools, and conferences where I could discuss my results with other specialists, and for guiding me during my own work on this thesis.

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GAMES.

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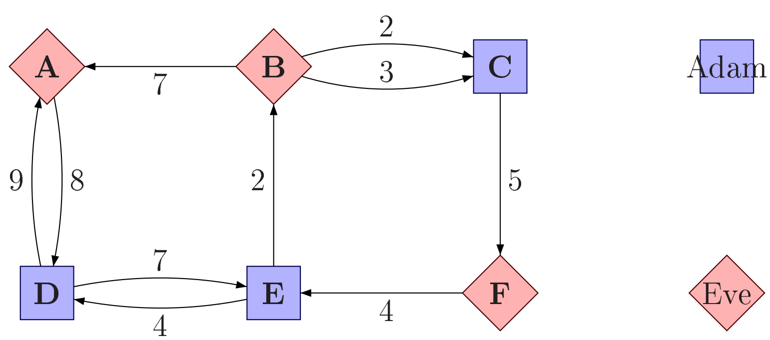
Chapter 2

Preliminaries

In this chapter we define all the basic notions we are working with. We start with an example of a game, then we define games, arenas, and winning conditions in general. Then we proceed to defining plays, strategies, and determinacy. We introduce determinacy types, like positional and half-positional determinacy. Finally, we show three types of arenas which appear in literature, and discuss how these types differ regarding positional strategies.

2.1 Example

Before giving the general definition of an infinite game, we show a typical example of a game.

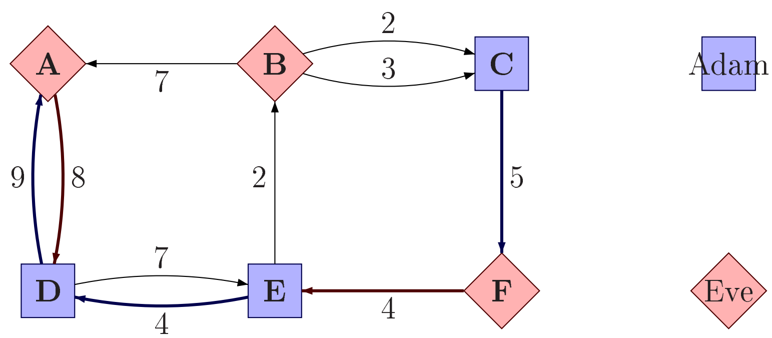


The picture above shows an *arena* the game is played on. The squares and diamonds are called *positions*; diamonds represents Eve's positions and squares represent Adam's positions.

The game starts by placing a token in one of the available positions. It can be either Eve's position or Adam's position. The owner chooses one of the moves (arrows) available from this position and moves the token to the position which is pointed to by the arrow. For example, if we start in **B**, Eve can choose either to go to **A** (which is also her position), or to Adam's position **C** (either by arrow labeled with 2, or by arrow labeled with 3). Now, this new position can again be either Eve's position or Adam's position — the owner decides the next move to be taken, and so on.

In this example, the play never ends: decisions made by both players define an infinite play. Now, there is a *winning condition* which says who will win, depending on the sequence of colors (i.e. labels) of moves which have been used during the infinite play.

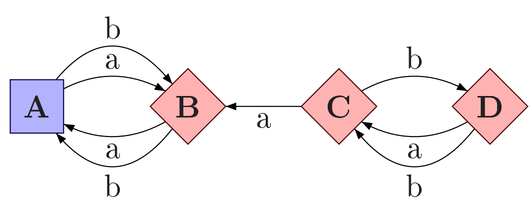
In the game above we could use the *parity condition*: Eve wins iff the greatest number appearing infinitely often is even. Otherwise, Adam is the winner.



By analyzing the game, we can find out that Adam has a winning strategy. In position **C**, always go to **F** (there is no other option anyway); Eve will have to go to **E**. In the position **E**, go to **D**, and in **D**, go to **A**. Now, Eve will have to return to **D**, as it is her only option. In position **D**, Adam always decides to go to **A**; thus, the sequence of colors (except the beginning) will be: 8, 9, 8, 9, ... and Adam will win.

Note that this strategy of Adam has the following property: in each position, always the same move is used. This is called a *positional* strategy.

Another example of a game follows. Now, Adam wants both letters **a** and **b** to appear infinitely often in the sequence of colors obtained from a play.



By analyzing the game, we get that Adam can win if the game starts in the positions **A** and **B** (an example winning strategy: when moving from **A** to **B**, he alternates between the two moves available, so he wins no matter what Eve is doing), and Eve can win if the game starts in **C** and **D** (in **C** she goes to **D** via **b**, and in **D** she goes to **C** also via **b**).

Note that Eve's winning strategy in **C** and **D** is positional, while Adam's winning strategy in **A** and **B** is not. That's what we mean by a *half-positional* game (or winning condition): from each position, either Eve has a positional winning strategy, or Adam has an arbitrary winning strategy.

2.2 Games

In this section we formally define games, arenas, and strategies.

We consider perfect information antagonistic infinite games played by two players, called conventionally Adam and Eve. Many names are used in literature (Alter and Ego, Abelard and Eloise, ...); if the players are not just named 0 and 1 (or I and II), usually they start with E and A, because they are associated with quantifiers $\exists$ (Eve) and $\forall$ (Adam).

Let C be a set of **colors** (possibly infinite). We use the standard notation and terminology from the theory of formal languages (or ω-languages) for finite and infinite sequences of colors. Thus, finite or infinite sequences of colors are sometimes called **words**, and sets of words are sometimes called **languages**. We sometimes identify colors with words of length 1, and words with languages with 1 element. $|w|$ is the length of word w, and $w_{|n}$ is the first n letters of the word w. ϵ is an empty word (of length 0). C^* and C^ω are the sets of all finite and infinite words over C, respectively. For two words $v \in C^*$ and $w \in C^* \cup C^\omega$, vw is a concatenation of v and w ($|vw| = |v| + |w|$). A word v is a **prefix** of a word w iff $w = vu$ for some u, and a **suffix** of w iff $w = uv$. For two languages L_1 and L_2 ($L_1 \subseteq C^*$), $L_1L_2 = \{vw : v \in L_1, w \in L_2\}$. For a language $L \subseteq C^*$, L^n is concatenation iterated n times: $L^0 = \{\epsilon\}, L^{n+1} = L^nL$, L^* is $\bigcup_{n\in\omega} L^n$. $\Pi_i w_i = w_1w_2w_3\ldots$ is an infinite concatenation, and $L^\omega = \{\Pi_i w_i : w_i \in L\}$.

An **arena** over C is a tuple $G = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$, where:

- Elements of $\mathrm{Pos} = \mathrm{Pos}_E \cup \mathrm{Pos}_A$ are called **positions**; Pos_A and Pos_E are disjoint sets of Adam's positions and Eve's positions, respectively.

- Elements of $\mathrm{Mov} \subseteq \mathrm{Pos} \times \mathrm{Pos} \times (C \cup \{\epsilon\})$ are called **moves**; (v_1, v_2, c) is a move from v_1 to v_2 colored by c. We denote $\mathrm{source}((v_1, v_2, c)) = v_1$, $\mathrm{target}((v_1, v_2, c)) = v_2$, $\mathrm{rank}((v_1, v_2, c)) = c$. We will write moves as $v_1 \xrightarrow{c} v_2$ instead of (v_1, v_2, c).

- ϵ denotes an empty word; a move $v \xrightarrow{\epsilon} w$ is viewed as colorless. However, there is a restriction on ϵ-moves: an arena is not allowed to contain infinite paths consisting only of them.

We say that an arena $G' = (\mathrm{Pos}'_A, \mathrm{Pos}'_E, \mathrm{Mov}')$ is a **subarena** of $G = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$ iff $\mathrm{Pos}'_A \subseteq \mathrm{Pos}_A$, $\mathrm{Pos}'_E \subseteq \mathrm{Pos}_E$, $\mathrm{Mov}' \subseteq \mathrm{Mov}$.

In our notation, Pos means the set of positions in the arena G. If another arena appears, say, G^0, then the set of positions in this game is denoted by either adding a respective index to Pos (say, Pos^0), or by treating Pos as an operator (say, $\mathrm{Pos}(G^0)$). Analogous notational convention is used for the sets Play and Win, which are defined later.

A game is a pair (G, W), where G is an arena, and W is a winning condition. A **winning condition** W over C is a subset of C^ω which is *prefix independent*, i.e., $u \in W \iff cu \in W$ for each $c \in C, u \in C^\omega$. We name specific winning conditions WA, WB, ...

Note that, contrary to some other works, when we consider winning conditions in this thesis, we mean prefix independent subsets of C^ω. Occasionally, we might use a game (G, W) where W is not prefix independent; we will then explicitly call W a *prefix dependent winning condition*.

As in the example above, the game (G, W) carries on in the following way. The play starts in some position v_1. The owner of v_1 (e.g. Eve if $v_1 \in \mathrm{Pos}_E$) chooses one of the moves leaving v_1, say $v_1 \xrightarrow{c_1} v_2$. If the player cannot choose because there are no moves leaving v_1, he or she loses. The next move is chosen by the owner of v_2; denote it by $v_2 \xrightarrow{c_2} v_3$. And so on: in the n-th move the owner of v_n chooses a move $v_n \xrightarrow{c_n} v_{n+1}$. If $c_1c_2c_3\ldots \in W$, Eve wins the infinite play; otherwise Adam wins.

A player can also resign instead of making a move; in this case, this player immediately loses. This option is used when there is no move possible from the current position; thus, each player immediately loses in his or her own position with no moves. The case when a player resigns is usually trivial, so there is no need to consider it in our proofs (resigning is never a winning move; a position with no moves corresponds to an $\exists$ or $\forall$ quantifier over an empty set).

A **play** in the arena G is a path in the arena graph. A play can be finite (the length of play $|\pi|$ is in ω) or infinite ($|\pi| = \omega$). We denote the set of all plays by Play, and $\mathrm{Play}_\infty, \mathrm{Play}_F, \mathrm{Play}_A, \mathrm{Play}_E \subseteq \mathrm{Play}$ are infinite plays, finite plays, and finite plays which end in Adam's and Eve's positions, respectively. We identify finite plays with (some) elements of $\mathrm{Pos} \cup \mathrm{Mov}^+$ (Pos represents plays which have just started and contain no moves yet, and Mov^+ are non-empty finite sequences of colors), and infinite plays with some elements of Mov^ω. Although plays are not exactly sequences of moves (since plays of length 0 are always in a specific position, and there is a restriction that the next move has to start where the previous one finished), we will sometimes use the same terminology and notation for them as for sequences, like prefix, suffix, concatenation, etc. By $\mathrm{source}(\pi)$ and $\mathrm{target}(\pi)$ we denote the initial and final position of the play, respectively (obviously infinite plays have no target). Thus, for a play of length 0 (we have just started in a position $\pi = v \in \mathrm{Pos}$) we have $\mathrm{source}(\pi) = \mathrm{target}(\pi) = v$, otherwise we have $\mathrm{source}(\pi) = \mathrm{source}(\pi_1)$, $\mathrm{target}(\pi_n) = \mathrm{source}(\pi_{n+1})$, and $\mathrm{target}(\pi_{|\pi|}) = \mathrm{target}(\pi)$.

2.3 Strategies

A **strategy for player** X (i.e. $X \in \{\mathrm{Eve}, \mathrm{Adam}\}$) is a partial function $s : \mathrm{Play}_X \to \mathrm{Mov}$. Intuitively, $s(\pi)$ for π ending in Pos_X says what X should do next. We say that a play π is **consistent** with strategy s for X if for each prefix π' of π such that $\pi' \in \mathrm{Play}_X$ the next move is given by $s(\pi')$, or $\pi' = \pi$ if $s(\pi')$ is not defined (i.e. the player X resigns).

A strategy s is **winning** (for X) from the position v if $s(\pi)$ is defined for each finite play π starting in v, consistent with s, and ending in Pos_X, and each infinite play starting in v consistent with s is winning for X. A strategy is winning from $M \subseteq \mathrm{Pos}$ iff it is winning from each $v \in M$.

A strategy s is **positional** if it depends only on $\mathrm{target}(\pi)$, i.e., for each finite play π we have $s(\pi) = s(\mathrm{target}(\pi))$.

Definition 2.1 *Let (G, W) be a game, and X be a player. The* **winning set** *of X, Win_X, is the set of positions from which X has a winning strategy.*

2.4 Determinacy

Definition 2.2 *A game is* **determined** *if for each position v one of the players has a winning strategy from v, i.e., $\mathrm{Win}_E \cup \mathrm{Win}_A = \mathrm{Pos}$.*

A game is **positionally determined** *iff for each position one of the players has a* positional *winning strategy from this position.*

A game is **half-positionally determined** *iff for each position either Eve has a* positional *winning strategy from this position, or Adam has (any) winning strategy from this position.*

A game is **co-half-positionally determined** *iff for each position either Adam has a* positional *winning strategy from this position, or Eve has (any) winning strategy from this position.*

A winning condition W is **determined, positional, (co-) half-positional** *iff for each arena G the game (G, W) is determined, positionally determined, (co-) half-positionally determined, respectively.*

A winning condition W is **finitely determined, positional, (co-) half-positional** *iff for each* finite *arena G the game (G, W) is determined, positionally determined, (co-)half-positionally determined, respectively.*

All games with a Borel winning condition are determined [Mar75], but there exist (exotic) games which are not determined.

We have introduced 8 classes of winning conditions (so far). Although in this thesis we focus on (finitely) half-positional winning conditions, several of our results can be stated and proven in a very similar way for each of these classes. To avoid repeating a similar result several times, we introduce the following notions.

Definition 2.3 *A* **basic arena type** *is a class of arenas γ such that if G is in γ and G' is a subarena of G, then G' is also in γ.*

Most of natural classes of arenas have this property, however, there are interesting arena types which are not basic, for example, arenas which are transition graphs of pushdown automata [Wal96, BSW03].

Definition 2.4 *A* **basic determinacy type** $D = (\alpha_E, \alpha_A, \gamma)$ *is given by three parameters:*

- α_E *— a class of admissible strategies for Eve (positional or arbitrary),*
- α_A *— a class of admissible strategies for Adam (positional or arbitrary),*
- γ *— a basic arena type.*

*We say that a strategy of player X is a D-***strategy** *iff it is in the class α_X. We say that an arena is a D-***arena** *iff it is in the class γ.*

*We say that a game (G, W) is D-***determined** *iff for every starting position one of the players has a D-strategy.*

*We say that a winning condition W is D-***determined** *if for every D-arena G the game (G, W) is D-determined.*

This definition encompasses all the classes of games and winning conditions mentioned in Definition 2.2. In particular, a winning condition is half-positional iff it is D-determined for $D =$ (positional, arbitrary, arbitrary).

Note that if a game (G, W) is $(\alpha_E, \alpha_A, \gamma)$-determined, then its dual game obtained by using the complement winning condition and switching the roles of players is $(\alpha_A, \alpha_E, \gamma)$-determined. Thus, W is $(\alpha_E, \alpha_A, \gamma)$-determined iff its complement is $(\alpha_A, \alpha_E, \gamma)$-determined.

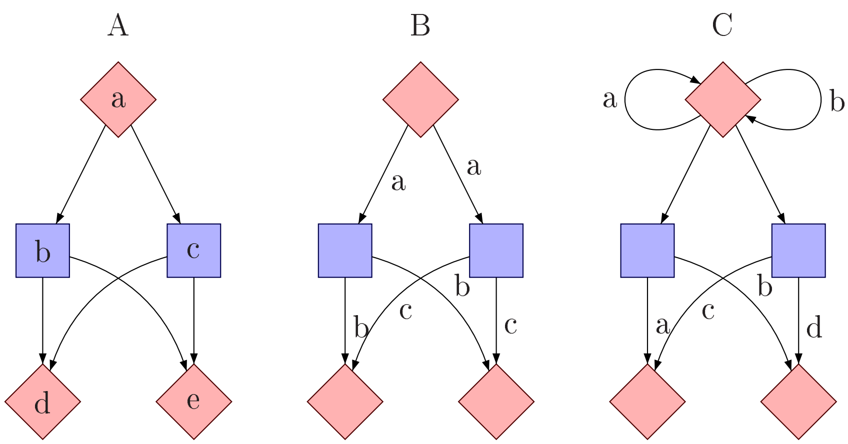
Sometimes, we will work with other classes of strategies than arbitrary and positional, and use an even more general definition.

Definition 2.5 *A* **determinacy type** $D = (\alpha_E, \alpha_A, \gamma)$ *is given by three parameters: classes of admissible arenas for both players* α_A *and* α_E*, and a class of arenas* γ*. D-strategies, D-arenas, D-determined games and winning conditions are defined similarly.*

2.5 Types of Arenas

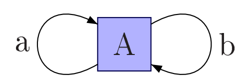
In the games defined above, the moves are colored, and it is allowed to have moves without colors. In the literature, several types of arenas are studied.

- ϵ-arenas (C), like the ones described above.
- *Move-colored arenas* (B). In this setting each move has a color assigned; moves labeled with ϵ are not allowed.
- *Position-colored arenas* (A). In this setting, colors are assigned to positions rather than to moves. Instead of $\mathrm{Mov} \subseteq \mathrm{Pos} \times \mathrm{Pos} \times C$ we have $\mathrm{Mov} \subseteq \mathrm{Pos} \times \mathrm{Pos}$ and a function $\mathrm{rank} : \mathrm{Pos} \to C$. As in (B), each position has a color assigned. The winner of a play in such games is defined similarly as for move-colored arenas.



If we take a position-colored arena and color each move p with the color $\mathrm{rank}(\mathrm{source}(p))$, we obtain an equivalent move-colored arena (this construction is illustrated on the picture). Therefore position-colored arenas are a subclass of move-colored arenas. Obviously, move-colored arenas are also a subclass of ϵ-colored arenas. When speaking about a determinacy type where we restrict arenas to position-colored or move-colored arenas, or we want to emphasize that we allow ϵ-arenas, we add the letter A, B or C (e.g. A-half-positional conditions when we restrict to position-colored arenas).

Hence C-half-positional conditions are a subclass of B-half-positional conditions, and B-half-positional conditions are a subclass of A-half-positional conditions. The inclusion between A-half-positional and B-half-positional conditions is proper: there is no way to transform a move-colored arena into a position-colored one such that nothing changes with respect to positional strategies (we can split a position into several new positions according to colors of moves which come into them, but then we obtain new positional strategies which were not positional previously). Indeed, we know examples of winning conditions which are A-positional but not B-positional. One of them is $C^*(\mathbf{ab})^*$, where $C = \{\mathbf{a}, \mathbf{b}\}$; for position-colored arenas we know from our current position to which color we should move next (when we are in position of color a, we should move to b, and vice versa), but not for edge-colored arenas, as is shown by the arena below. (We don't give full proofs, since we don't have introduced necessary techniques yet; a full proof is given later for example 8.19, which is based on the same idea.) Another example is min-parity [GW06]. B-positional determinacy has been characterized in [CN06]; this result can be easily generalized to ϵ-arenas. Positional determinacy on ϵ-arenas has been studied in [Zie98].



The question whether the inclusion between C-half-positional conditions and B-half-positional conditions is proper remains open. (However, Example 8.19 in Section 8.5 about persistent strategies presents a winning condition which admits positional strategies for A-arenas, only persistent strategies (Definition 8.17) for B-arenas, but not even persistent strategies for C-arenas; thus, for persistent strategies the inclusion is proper.)

Note that, when considering half-positional determinacy of winning conditions on arenas with ϵ labels, there is no difference whether we label positions or moves. Indeed, for each move-colored ϵ-arena, if we replace each move $v_1 \to v_2$ colored with c by $v_1 \to v \to v_2$, color v with c, and leave all the original positions (i.e., v_1, v_2 etc.) colorless, we obtain an equivalent position-colored ϵ-arena — strategies in one arena can be interpreted in the other one.

In this thesis, we concentrate on ϵ-arenas since we think that this class gives the least restriction on arenas. As the example above, $C^*(\mathbf{ab})^*$, suggests, positional strategies for move-colored games are "more memoryless" than for position-colored games since they do not even remember the last color used, although winning conditions for position-colored games (like min-parity) may also be interesting. As we will see in the sequel, allowing our arenas to contain ϵ-moves — despite potential greater generality of such arenas — usually does not make our proofs harder, and sometimes even makes them easier and more natural.

2.6 Extensions

In some papers a more general situation is investigated, where instead of a winning condition we have a **payoff mapping** $u : C^\omega \to \mathbb{R}$. In such games Eve's and Adam's goals are respectively maximization and minimization of $u(c_1c_2c_3\ldots)$. The payoff mapping can be intuitively interpreted as the quantity of money which Eve wins from Adam. Payoff mapping is a generalization of the winning condition (we can get the equivalent payoff mapping by taking the characteristic function of a winning condition).

Chapter 3

Basic Tools

In this chapter we present our basic tools and the most important positional winning conditions. In the first section we prove some well known properties of positional (and also not necessarily positional) strategies in games with prefix independent winning conditions. In the next section, we use them to prove Lemma 3.5 which will be used in many proofs of half-positional determinacy of various winning conditions. Then, we present Büchi and co-Büchi conditions, and a closure property regarding them (Theorem 3.7).

In the last section we show how our results can be used to immediately give an alternative proof for positional determinacy of the parity condition. We also cite and generalize some interesting facts regarding parity conditions.

3.1 Naturalness of Determinacy Types

In this section we will show some well known basic properties and definitions which apply to strategies in games with prefix independent winning conditions. Prefix independence of W is very important for these properties. Although they are of most interest for positional strategies, they are true for arbitrary ones (i.e., not necessarily positional) too, so we prove them in a general way, for all basic determinacy types (see page 16).

Definition 3.1 *Let* $G = (\mathrm{Pos}_E, \mathrm{Pos}_A, \mathrm{Mov})$ *be an arena, and* X *be a player. For* $M \subseteq \mathrm{Pos}$*, let* $\mathrm{Next}_X(M)$ *be the set of all* X*'s positions from which at least one move reaches* M*, and all opponent's positions from which all moves reach* X*. Let* $\mathrm{Attr}_X(N)$ *be the least* $M \subseteq \mathrm{Pos}$ *(with respect to inclusion) such that* $M \supseteq N$ *and* $M \supseteq \mathrm{Next}_X(M)$*.*

Intuitively, $\mathrm{Attr}_X(M)$ ("attractor") is a set of positions from which X has a strategy to reach M. It can be obtained as the least fixpoint of the operator $\mathrm{Next}^?_\mathrm{X}(\mathrm{M}) = \mathrm{M} \cup \mathrm{Next}_\mathrm{X}(\mathrm{M})$ which contains the set N.

Definition 3.2 *Let $G = (\mathrm{Pos}_E, \mathrm{Pos}_A, \mathrm{Mov})$ be an arena, and X be a player. Let $M \subseteq \mathrm{Pos}$, and s be a strategy for X. Then $M[s]$ is the set of all positions which occur in some play starting from M and consistent with s.*

Theorem 3.3 *Each basic determinacy type D has the following properties for each arena G, player X, and winning condition W:*

- *(forward) If X has a winning D-strategy s from M, then X has a winning D-strategy from $M[s]$.*
- *(backward) If X has a winning D-strategy s from M, then X has a winning D-strategy from $\mathrm{Attr}_X(M)$.*
- *(globalization) Let S be a set of D-strategies for X such that each $s \in S$ is winning from $U(s) \subseteq \mathrm{Pos}$. Then X has a winning D-strategy from $\bigcup_{s\in S} U(s)$.*
- *(excision) Let s be a winning D-strategy from M for X, and $M = \mathrm{Attr}_X(M) = M[s]$. Let G' be the game obtained by removing all the positions in M. Then if a player Y (either X or opponent) has a winning D-strategy from a set M' in the game (G', W), then Y also has a winning D-strategy from M' in G.*

Definition 3.4 *We say that a determinacy type D is* **natural** *if it has all properties from Theorem 3.3.*

Proof of Theorem 3.3 The forward condition is obvious from prefix independence.

To prove the globalization condition, assume that S is well ordered, $S = \{s_\alpha\}_{\alpha<\gamma}$. Since the forward condition is satisfied, we can assume without loss of generality that $U(s_\alpha) = U(s_\alpha)[s_\alpha]$ (if this is not satisfied, let $U = U(s_\alpha)[s_\alpha] \neq U(s_\alpha)$; from forward condition we know that there is a strategy s' which is winning in U; we replace s_α with s' and let $U(s') = U$). The strategy s winning from $\bigcup_\alpha U(s_\alpha)$ is as follows. Let $\pi \in \mathrm{Play}_X$. Let α be the smallest ordinal for which $\mathrm{target}(\pi) \in U(s_\alpha)$. Let π' be the longest suffix of π for which $\mathrm{source}(\pi')$ is also in $U(s_\alpha)$. Then $s(\pi) = s_\alpha(\pi')$.

We will show that s is indeed winning. Let $\pi \in \mathrm{Play}_\infty$ be consistent with s. Let α_n be α which was used for the finite prefix $\pi_{|n}$ (i.e. after the nth move). Since $U(s_\alpha) = U(s_\alpha)[s_\alpha]$, our strategy never leaves $U(s_{\alpha_n})$ in the nth move, and thus α_n is a non-increasing sequence. Hence, there exists a m such that $\forall n \geq m\ \alpha_n = \alpha_m$. Since our strategy, except the first m moves, plays consistently with s_{α_m}, and W is prefix independent, X wins the play π.

To prove the backward condition, we can use the forward condition to assume that $M = M[s]$. Note that if X has a winning D-strategy from M, then X has a winning D-strategy from $\mathrm{Next}^?_X(M) = M \cup \mathrm{Next}_X(M)$. Indeed, if the position $v \in (\mathrm{Next}_X(M) - M) \cap \mathrm{Pos}_E$, the strategy is to use the move which witnesses $v \in \mathrm{Next}_X(M)$, and then to use Eve's strategy in M. In $v \in (\mathrm{Next}_X(M) - M) \cap \mathrm{Pos}_A$, just let Adam do a move and continue using our strategy in M.

The least fix point $\mathrm{Attr}_X(M)$ can be obtained by iterating $\mathrm{Next}^?_X(M)$ (possibly requiring a transfinite number of iterations). Thus, by iterating, we obtain that X has a winning strategy in $\mathrm{Attr}_X(M)$ (using e.g. the globalization condition for transfinite steps).

To show the excision condition, we have to find the strategy from M' in the original arena G. The strategy is to use s' until Y's opponent decides to leave G' — i.e. enter M. Since we assumed that $M = \mathrm{Attr}_X(M)$, this is possible only for $X = Y$. In this case, Y also has a winning strategy s in M, which he or she can use. ∎

3.2 An Useful Lemma

Lemma 3.5 *Let D be a natural determinacy type. Let $W \subseteq C^\omega$ be a winning condition. Suppose that, for each non-empty D-arena G over C, there exists a non-empty subset $M \subseteq \mathrm{Pos}_G$ such that in game (G, W) one of the players has a D-strategy winning from M. Then W is D-determined.*

Equivalently, instead of taking a non-empty subset M, we could say that there exists a position $v \in \mathrm{Pos}_G$ such that in game (G, W) one of the players has a D-strategy winning from v. Although that wording might be simpler to understand, we will use the wording above, since that is how our lemma will be used. Actually, when we use our lemma to show half-positional determinacy, we will usually show that either Adam has a winning strategy everywhere, or Eve has a positional winning strategy in a non-empty subset.

Proof of Lemma 3.5 Let $G = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$ be a D-arena.

The idea of the proof is as follows. From our hypothesis we know that we can determine the winner and his D-strategy in some positions in G. We remove these positions from G and we use our hypothesis again, determining the winner in some other set of positions. We iterate (possibly needing a transfinite number of iterations) until we remain with an empty set. When done correctly, this leads to determining the winner in the whole G, together with D-strategies in (G, W).

We will define (possibly transfinite) sequences $P_\alpha \subseteq \text{Pos}$, $M_\alpha \subseteq \text{Pos}$, G_α of subarenas, X_α of players, and s_α of strategies in the following way.

Let $P_0 = \text{Pos}$. The sequences end when $P_\alpha = \emptyset$. Otherwise, let $G_\alpha = (\text{Pos}_A \cap P_\alpha, \text{Pos}_E \cap P_\alpha, \text{Mov} \cap P_\alpha \times P_\alpha \times C)$. From our hypothesis we know that there exists a player X_α and a subset $M_\alpha \subseteq P_\alpha$ such that X_α has a winning D-strategy s_α in G_α from M_α. (Of course, there can be many possible choices of M_α, X_α and s_α — we can choose any one of them). Without loss of generality we can assume that $M_\alpha = M_\alpha[s_\alpha]$ and $M_\alpha = \text{Attr}_X(M_\alpha)$ (we use forward and backward conditions to fix M_α and s_α in case if it is not true). Also let $P_{\alpha+1} = P_\alpha - M_\alpha$, and for a limit ordinal λ, let $P_\lambda = \bigcap_{\alpha<\lambda} P_\alpha$.

Let Y be any of the players. We will construct the sequence of Y's D-strategies s'_α, such that s'_α is winning from $\bigcup_{\gamma<\alpha:X_\gamma=Y} M_\gamma$. For a limit ordinal α, s'_α can be obtained from s'_γ, for $\gamma < \alpha$, by the globalization condition. Otherwise, we obtain $s'_{\alpha+1}$ using the excision and globalization conditions on s'_α and s_α.

This sequence of strategies ends with $s_Y = s'_\beta$. Thus, for each player Y we have found a D-strategy s_Y winning from $M_Y = \bigcup_{\gamma<\beta:X_\gamma=Y} M_\gamma$. We have $M_A \cup M_E = \bigcup_{\gamma<\beta} M_\gamma = \text{Pos}$, hence the game is D-determined. ■

3.3 Büchi and Co-Büchi Conditions

Definition 3.6 *For $S \subseteq C$, WB_S is the set of infinite words where elements of S occur infinitely often, i.e. $(C^*S)^\omega$. Winning conditions of this form are called* Büchi conditions. *Complements of Büchi conditions, $WB'_S = C^*(C-S)^\omega$ are called* co-Büchi conditions.

Theorem 3.7 *Let D be a basic determinacy type. Let $W \subseteq C^\omega$ be a winning condition, and $S \subseteq C$. If W is D-determined, so is $W \cup WB_S$.*

Proof of Theorem 3.7 We will show that the assumption of Lemma 3.5 holds. Let our arena be $G = (\text{Pos}_E, \text{Pos}_A, \text{Mov})$. *$S$-moves* are moves p such that $\text{rank}(p) \in S$.

Let G' be G with a new position $\top$ added. The position $\top$ belongs to Adam and has no outgoing moves, hence Adam loses here. For each S-move p we change $\text{target}(p)$ to $\top$.

Since Adam immediately loses after doing an S-move in G', the winning conditions W and $W \cup WB_S$ are equivalent for G', i.e. a play is winning in the game (G', W) iff it is winning in the game $(G', W \cup WB_S)$. Thus, a strategy is winning in (G', W) iff it is winning in $(G', W \cup WB_S)$, and we can use D-determinacy of W to find the winning sets $\mathrm{Win}'_E, \mathrm{Win}'_A$ and winning D-strategies s'_E, s'_A in G'.

Suppose $\mathrm{Win}'_A \neq \emptyset$. We can see that since Adam's strategy wins in G' from a starting position in Win'_A, he also wins in G from there by using the same strategy (the game G' is "harder" for Adam than G). Thus the assumption of 3.5 holds (we take $M = \mathrm{Win}'_A$).

Now suppose that $\mathrm{Win}'_A = \emptyset$. We will show that in the game G Eve has a winning D-strategy s in Pos everywhere, hence the assumption of Lemma 3.5 holds as well (we take $M = \mathrm{Pos}$).

The strategy is as follows. For a finite play π we take $s(\pi) = s_E(\pi')$, where π' is the longest final segment without any S-moves, unless when $s_E(\pi')$ is a move to $\top$. In this case, there had to be at least one S-move from $\mathrm{target}(\pi')$ in G, and Eve makes one of them.

The strategy s is positional if s_E is positional. Let π be a play consistent with s. There are two possibilities: there is either finite or infinite number of S-moves in π. If the number is infinite, then Eve wins (as she wins WB_S). If the number is finite, then $\pi = \pi_0\pi'$, where π_0 ends with the last S-move (possibly π_0 is empty). Hence, π' does not contain any S-moves and is consistent with s_E, thus Eve also wins π', and also π because of prefix independence. Therefore, s is indeed a winning D-strategy. ■

Note that, by duality, Theorem 3.7 shows that if W is D-determined, then so is $W \cap WB'_S$.

Although this proof works for all basic determinacy types, there are natural generalized determinacy types for which it fails. Indeed, the determinacy type of positional/suspendable winning conditions (see page 61 later) is natural, but the claim of Theorem 3.7 is false for them. On the other hand, in Section 8.5 about persistent strategies later we present natural determinacy types for which the claim of Theorem 3.7 is true, although it has to be proven in a different way.

3.4 Parity Conditions

The **parity condition** of rank n is the winning condition over the set of colors $C = \{0, 1, \ldots, n\}$ defined with

$$WP_n = \{w \in C^\omega : \limsup_{i\to\infty} w_i \text{ is even}\}. \tag{3.1}$$

This is one of the most important classical winning conditions. Many proofs of its positional determinacy are already known. Theorem 3.7 immediately gives yet another one: it is enough to start with an empty winning condition (which is positionally determined) and apply Theorem 3.7 and its dual n times.

It is worth to remark that in case of infinite arenas the parity conditions are the only ones which admit positional determinacy.

Theorem 3.8 *Let $W \subseteq C^\omega$ be a winning condition. The following properties are equivalent:*

1. $W = h^{-1}(WP_n)$ *for some* $h : C \to \{0, 1, \dots, n\}$, *where by* $h(w)$ *for* $w \in C^\omega$ *we mean the word* v *such that* $v_n = h(w_n)$ *(we call such a* W *a* generalized *parity condition);*

2. W *is positionally determined;*

3. (G, W) *is positionally determined for each arena* G *over* C *where either* $\mathrm{Pos}_E = \emptyset$ *or* $\mathrm{Pos}_A = \emptyset$*;*

4. *Let* $W_f = \{u \in C^+ | u^\omega \in W\}$*. We have* $W_f^\omega \subseteq W$ *and* $(C^+ - W_f)^\omega \subseteq C^\omega - W$*.*

The equivalence of (1) and (2) has been shown in [CN06]. Note that this theorem works only in case of edge-colored arenas (B) and ϵ-arenas (C), not position-colored arenas (see Section 2.5 for definitions of arena types, and examples of A-positional winning conditions).

Proof

1→2 is a simple generalization of a well known fact — namely, positional determinacy of parity games ([Mos91], [EJ91], [McN93]). As mentioned above, it can be also shown by applying Theorem 3.7 and its dual n times.

2→3 is obvious (a special case).

2→4 is proven in [CN06] (as Lemma 7). Actually, only one-player arenas are used in the proof, so we get $3 \to 4$.

2→1 is proven in [CN06]. However, the assumption (2) is never used except the proof of Lemma 7 (i.e., implication 2→4) and Lemma 9. So, to show $4 \to 1$, we only have to prove Lemma 9 using condition (4)[1].

Lemma 3.9 (Lemma 9 from [CN06]) *Assume that Condition (4) from Theorem 3.8 is true. Then for any* $L, L' \subseteq C^+$ *we have*

$$\forall v \in L' \ \exists u \in L \ uv \in W_f \text{ iff } \exists u \in L \forall v \in L' uv \in W_f$$

[1] The fact that Lemma 9 is a consequence of condition (4) has been noticed by Hugo Gimbert.

Proof of Lemma 3.9 (←) is obvious. To prove (→), assume to the contrary that for each $u \in L$ there exists $v \in L'$ such that $uv \notin W_f$. We define sequences $v_n \in L'$ and $u_n \in L$ by induction. Let u_1 be any element of L. Let $v_n \in L'$ be such that $u_n v_n \notin W_f$. Let u_{n+1} be such that $v_n u_{n+1} \in W_f$. The word $v_1 u_2 v_2 u_3 \ldots \in W_f^\omega \subseteq W$ (by (4)). On the other hand, the word $u_1 v_1 u_2 v_2 \ldots \in C^+ - {W_f}^\omega \subseteq C^\omega - W$ (by dual in (4)). This is a contradiction, since W is prefix independent. ■

In the case of finite arenas there are more positional winning conditions, and we don't have neither 2→4 nor 2→1. For example, the winning condition $WF(A)$ from Section 5 below, where A and its complement are both convex sets, is finitely positional. However, we have equivalence of (2) and (3) (a very elegant result from [GZ05]).

Chapter 4

Concave Winning Conditions

In the following chapters, we give some examples of half-positionally determined winning conditions. We start by giving a simple combinatorial property which guarantees finite half-positional determinacy.

4.1 Definition

Definition 4.1 *A word $w \in C^* \cup C^\omega$ is a* **shuffle** *of words w_1 and w_2, iff for some sequence of words (u_n), $u_n \in C^*$*

- $w = \prod_{k\in\mathbb{N}} u_k = u_0u_1u_2u_3u_4u_5u_6u_7u_8\ldots$,
- $w_1 = \prod_{k\in\mathbb{N}} u_{2k+1} = u_1u_3u_5u_7\ldots$,
- $w_2 = \prod_{k\in\mathbb{N}} u_{2k} = u_0u_2u_4u_6\ldots$.

Definition 4.2 *A winning condition W is* **convex** *if as a subset of C^ω it is closed under shuffles, and* **concave** *if its complement is convex.*

Example 4.3 *Parity conditions (including Büchi and co-Büchi conditions) are both convex and concave.*

Proposition 4.4 *Concave winning conditions are closed under union. Convex winning conditions are closed under intersection.*

Example 4.5 *Let $C = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. The winning condition $WB'_{\{\mathbf{a}\}} \cup WB'_{\{\mathbf{b}\}}$ (co-Büchi condition (Definition 3.6); in other words, Eve wins iff at least one of letters* **a** *and* **b** *appears finitely often) is concave, but not convex.*

Example 4.6 *Let C be an infinite set. The following winning conditions are both convex and concave:*

- Exploration condition*: the set of all v in C^ω such that $\{v_n : n \in \omega\}$ is infinite.*
- Unboundedness condition*: the set of all v in C^ω such that no color appears infinitely often.*

Decidability and positional determinacy of these conditions on (infinite) pushdown arenas where each position has a distinct color has been studied in [Gim04] (exploration condition) and [BSW03], [CDT02] (unboundedness condition).

Another example, which justifies the names *convex* and *concave*, is given in Chapter 5 below.

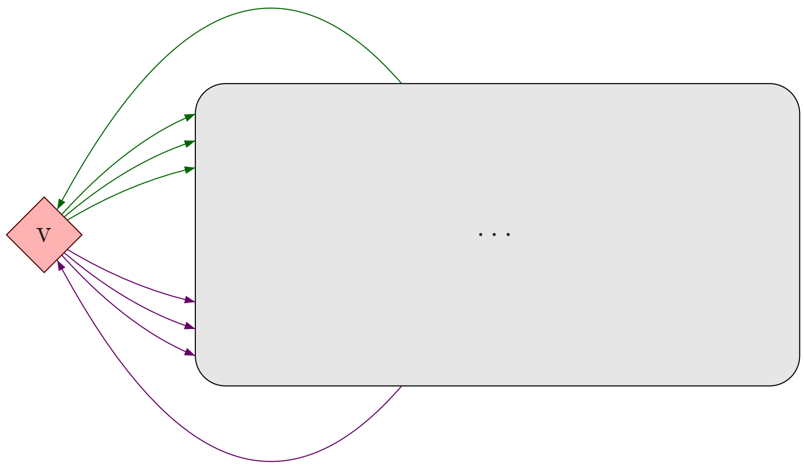
4.2 Half-positional Determinacy

Theorem 4.7 *Concave winning conditions are half-positionally finitely determined.*

The proof goes by induction over Mov, and is based on the following idea. Let v be Eve's position, with outgoing moves $p_1, p_2, \ldots$. Suppose that Eve cannot win by using only one of these moves. Then, since the winning condition is concave, she also cannot win by using many of these moves — because it can be written as a shuffle of subplays that appear after each move $p_1, p_2, \ldots$, and Adam wins all of these plays.

Proof of Theorem 4.7 Let $W \subseteq C^\omega$ be a concave winning condition in the game (G, W), where $G = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$. A proof by induction on $|\mathrm{Mov}|$.

Let v be a position belonging to Eve, where she has more than one move. If there are no such positions, the game (G, W) must be half-positionally determined from definition.



Let M be a set of Eve's possible moves from v. Let $M = M_1 \cup M_2$, where M_1 and M_2 are non-empty and disjoint. Let $G^i = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov} - M_{3-i})$, $G^A = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov} - M)$.

From the induction hypothesis we know that the games (G^1, W), (G^2, W) and (G^A, W) are half-positionally determined. Let Win_E^i and Win_A^i be winning sets for Eve and Adam, respectively, in the games (G^i, W) for $i \in \{1, 2, A\}$,and let s_i and t_i be the winning strategies of Eve in Win_E^i and Adam in Win_A^i, respectively, in these games. Suppose s_i is a positional strategy for $i \in \{1, 2, A\}$.

First, assume that $v \in \mathrm{Win}_i^E$ for some i. In this case the strategy s_i is also winning for Eve in the set Win_i^E in the arena G (since the only difference between G_i and G is that Eve has more possibilities in G). On the other hand, t_i is a winning strategy for Adam in the set Win_i^A in the arena G, since each play consistent with t_i is winning for Adam and therefore must not go through v (by prefix independence, Eve would win otherwise), hence Eve is unable to use her additional possibilities.

Now, assume that $v \in \mathrm{Win}_1^A$ and $v \in \mathrm{Win}_2^A$. Since $v \in \mathrm{Win}_i^A$, Adam is able to win each play in G_i which goes through v. Therefore the winning sets in G_i are the same as in G_A (again, prefix independence). Therefore, if $v \in \mathrm{Win}_1^A$ and $v \in \mathrm{Win}_2^A$, we have $\mathrm{Win}_1^A = \mathrm{Win}_2^A$ (since both of them are equal to Win_A^A) and $\mathrm{Win}_1^E = \mathrm{Win}_2^E$.

Similarly to Adam's strategy in the first case, Eve's (positional) strategy s_1 remains winning for Eve in the set Win_1^E in the game G. We will show a winning strategy for Adam in the set Win_1^A.

Let $\pi = \pi_1 \ldots \pi_m$ be a finite play. We will present π as a shuffle of two plays $\pi_{(1)}$ and $\pi_{(2)}$, where $\pi_{(i)}$ is a play in G_i.

Let $K = \mathrm{dom}\ \pi = \{1, \ldots, m\}$. Let $S_v = \{k \in K : \mathrm{source}(\pi_k) = v\}$. We define the function $f : K \to \{1, 2\}$ in the following way. If $k < \min S_v$, we take $f(k) = 1$. Otherwise, $f(k) = i$ iff $\pi_{k'} \in M_i$, where k' is the greatest element of S_v such that $k' \leq k$.

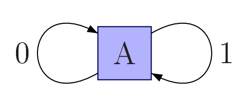
For $i = 1, 2$, let $\pi_{(i)} = \Pi_{k \in K} \pi_k^{[f(k)=i]}$, where $w^{[\phi]}$ denotes w if ϕ is true, and the empty word ϵ otherwise. One can easily see that π, as a word over Mov, is then a shuffle of $\pi_{(1)}$ and $\pi_{(2)}$.

It can be easily checked that $\pi_{(i)}$ is a play. For $j = f(m)$ we have $\mathrm{target}(\pi_{(j)}) = \mathrm{target}(\pi)$. Let $t(\pi) = t_j(\pi_{(j)})$. If Adam consistently plays with the strategy t, the plays $\pi_{(i)}$ are consistent with t_i for $i = 1, 2$.

We check that t is indeed a winning strategy for Adam in the set Win_1^A in the game (G, W). Let π be an infinite play consistent with t. Like for finite plays, π is a shuffle of $\pi_{(1)}$ and $\pi_{(2)}$. Hence $\mathrm{rank}(\pi)$, the sequence of colors in the play π, is a shuffle of $\mathrm{rank}(\pi_{(1)})$ and $\mathrm{rank}(\pi_{(2)})$. The plays $\pi_{(i)}$ for $i = 1, 2$ are either finite or winning for Adam (as they are consistent with t_i). If $\pi_{(i)}$ is finite, $\pi_{(3-i)}$ is infinite and winning for Adam; from prefix independence of W we get that π is also winning for Adam. If both plays are infinite, $\mathrm{rank}(\pi_{(1)}) \notin W$ and $\mathrm{rank}(\pi_{(2)}) \notin W$; from concavity of W we get that also $\mathrm{rank}(\pi) \notin W$. ■

This theorem gives yet another proof of finite positional determinacy of parity games, and also finite half-positional determinacy of unions of families of parity conditions (where each parity condition may use a different rank for a given color). Half-positional determinacy of Rabin conditions (finite unions of parity conditions) over infinite arenas has been proven in [Kla92] (see also [Gra04], and Theorem 7.12 in this thesis).

Note that, in general, concavity does not imply half-positional determinacy over infinite arenas — for examples see Chapter 5 below, and also Example 4.6 and Theorem 7.2. Also, half-positional determinacy (even over infinite arenas) does not imply concavity — examples can be found in Chapters 5 and Section 6.2 (Proposition 6.7 and the note above it). These two facts are especially visible in the table in Section 5.5, which compares (among others) two very similar winning conditions, one of which is concave but not (infinitely) half-positional, while the other is infinitely half-positional but not concave (only weakly).



Concavity does not force any bound on the memory required by Adam. Indeed, let $x \in [0, 1] - \mathbb{Q}$, $C = \{0, 1\}$, and consider the game (G, W), where G is the arena with one Adam's position A and two moves $A \to A$ colored 0 and 1 respectively, and let W be the set of sequences (c_n) such that $\sum_{i=1}^{n} c_i/n$ is not convergent to x. This winning condition is concave (Theorem 5.1 in Chapter 5 below), but Adam obviously requires unbounded memory here.

A related property has been shown in [MT02]: a winning condition W is called *positive* iff its complement is closed under supersequences (i.e., shuffles with C^ω). Theorem 3 from [MT02] says that games with positive winning conditions admit persistent winning strategies for Eve. A winning strategy s is *persistent* iff $s(\pi_1)$ equals $s(\pi_1\pi_2)$ whenever $\mathrm{target}(\pi_1) = \mathrm{target}(\pi_1\pi_2)$ (i.e., Eve always chooses the same move from each position, but she can decide which move she takes not before game, but when the game enters this position). Positiveness is a stronger property than concavity (for example, the parity condition is concave, but not positive), and persistence is a weaker property than positionality; however, we are not limited to finite arenas (persistent strategies are not interesting on finite arenas, see Corollary 8.23 later). There will be more about persistent strategies in Section 8.5 later.

4.3 Weakening the Concavity Condition

In [GZ04] a result similar to Theorem 4.7 has been obtained in the case of full positional determinacy. To present it, we need the following definition:

Definition 4.8 *A winning condition W is* **weakly convex** *iff for each sequence of words (u_n), $u_n \in C^*$, if*

1. $u_1u_3u_5u_7\ldots \in W$,
2. $u_2u_4u_6u_8\ldots \in W$,
3. ($\star$) $\forall i\ (u_i)^\omega \in W$,

then $u_1u_2u_3u_4\ldots \in W$.

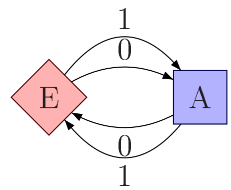
A winning condition W is **weakly concave** *iff its complement is weakly convex.*

In the case of normal convexity there is no ($\star$).

[GZ04] defines *fairly mixing* payoff mappings; in the case of prefix independent winning conditions *fairly mixing* resolves to the conjunction of weak concavity and weak convexity. Theorem 1 from [GZ04] says that games on finite arenas with fairly mixing payoff mappings are positionally determined.

Unfortunately, weak concavity is not enough for half-positional finite determinacy.

Proposition 4.9 *There exists a weakly concave winning condition, WQ, which is not half-positionally finitely determined.*



Proof Let $C = \{0, 1\}$. For $w \in C^\omega$ let $P_n(w)$ be the number of 1's among the first n letters of w, divided by n. The winning condition WQ is a set of w such that $P_n(w)$ is convergent and its limit is rational. It can be easily seen that for each $u \in C^+$ we have $u^\omega \in WQ$. Therefore $(\star)$ is never satisfied for the complement of WQ, hence WQ is a weakly concave winning condition. However, WQ is not half-positionally determined. Consider the arena with two positions $E \in \mathrm{Pos}_E$, $A \in \mathrm{Pos}_A$, and moves $E \xrightarrow{0} A$, $E \xrightarrow{1} A$, $A \xrightarrow{0} E$ and $A \xrightarrow{1} E$. If Eve always moves in the same way, Adam can choose the moves 0 and 1 in an irrational proportion, ensuring his victory. However, Eve wins by always moving with the color opposite to Adam's last move — the limit of $P_n(w)$ is then $1/2$. ■

Note that the given WQ satisfies the even stronger condition obtained by replacing $\forall i$ by $\exists i$ in $(\star)$ in Definition 4.8.

Chapter 5

Geometrical Conditions

In this chapter we show some half-positional determinacy results for *geometrical conditions*, which are based on the ideas similar to that used by the *mean payoff game* (sometimes called *Ehrenfeucht-Mycielski game*). We also show the relations between geometrical conditions and concave winning conditions.

5.1 Definition

Let $C = [0,1]^d$ (where $[0,1]$ is the real interval; we can also use any compact and convex subset of a normed space). For a word $w \in C^+$, let $P(w)$ be the average color of w, i.e., $\frac{1}{|w|}\sum_{k=1}^{|w|} w_k$. For a word $w \in C^\omega$, let $P_n(w) = P(w_{|n})$ ($w_{|n}$ — an n-letter prefix of w).

Let $A \subseteq C$. We want to construct a winning condition W such that $w \in W$ whenever the limit of $P_n(w)$ belongs to A. Since not every sequence has a limit, we have to define the winner for all other sequences.

Let $WF(A)$ be a set of w such that each cluster point of $P_n(w)$ is an element of A. Let $WF'(A)$ be a set of w such that at least one cluster point of $P_n(w)$ is an element of A. Note that $WF'(A) = C^\omega - WF(C - A)$.

As we will see, for half-positional determinacy the important property of A is whether the complement of A is convex — we will call such sets A *co-convex* (as *concave* usually means "non-convex" in geometry).

Geometrical conditions have a connection with the *mean payoff game*, whose finite positional determinacy has been proven in [EM79]. In the mean payoff game, C is a segment in $\mathbb{R}$ and the payoff mapping is $u(w) = \liminf_{n\to\infty} P_n(w)$. If $A = \{x : x \geq x_0\}$ then $u^{-1}(A)$ ("Eve wants x_0 or more") is exactly the geometrical condition $WF(A)$. Of course, the dual payoff, defined with $u(w) = \limsup_{n\to\infty} P_n(w)$, corresponds to $WF'(A)$. (In case of finite arenas it does not matter whether we take lim sup or lim inf, since if both players use optimal strategies, the sequence $P_n(w)$ will be convergent. However, things change for infinite arenas.)

Geometrical conditions are a generalization of such winning conditions to a larger class of sets A and C.

5.2 Concave and Convex

In this section we show how notions of convexity and concavity of winning conditions, introduced in Chapter 4 (Definition 4.2), are related to geometrical convexity of the set A.

Theorem 5.1 *We have:*

1. *$WF'(A)$ is weakly convex iff A is a closed convex subset of C.*
2. *$WF'(A)$ is convex iff A is a trivial subset of C (i.e., $A = \emptyset$ or $A = C$).*
3. *$WF'(A)$ is weakly concave iff A is a co-convex subset of C.*
4. *$WF'(A)$ is concave iff A is a co-convex subset of C.*
5. *$WF(A)$ is weakly convex iff A is a convex subset of C.*
6. *$WF(A)$ is convex iff A is a convex subset of C.*
7. *$WF(A)$ is weakly concave iff A is an open co-convex subset of C.*
8. *$WF(A)$ is concave iff A is a trivial subset of C.*

To prove it, we need the following lemmas:

Lemma 5.2 *If A is a convex subset of C then $WF(A)$ is convex.*

Proof

Now, suppose A is convex; we will show that $WF(A)$ is convex.

Let w_3 be a shuffle of w_1 and w_2, where $w_1, w_2 \in WF(A)$. Let B_k, for $k = 1,\ 2$, be a set of cluster points of $P_n(w_k)$, and B_3 be the convex hull of $B_1 \cup B_2$. Since $B_1 \subseteq A$ and $B_2 \subseteq A$, also $B_3 \subseteq A$. All the sets B_1, B_2, B_3 are compact. Let δ_n^k be the distance of $P_n(w_k)$ from the set B_k for $k = 1, 2, 3$. The sequence (δ_n^k) converges to 0 for $k = 1, 2$. We will show that (δ_n^3) also converges to 0.

Let $\epsilon > 0$. Let N be a number such that for all $n \geq N$ we have $\delta^1_n < \epsilon$ and $\delta^2_n < \epsilon$. Let $n > ND/\epsilon$, where D is the diameter of C, i.e., the maximum distance between two colors. The word $w_{3|n}$ is a shuffle of w_{1m} and $w_{2|m'}$ for some $m + m' = n$. One can easily show the following:

$$P_n(w_0) = \frac{m}{n} P_m(w_1) + \frac{m'}{n} P_{m'}(w_2). \tag{5.1}$$

For $k = 1, 2$, let $P_m(w_k) = b_k + x_k$, where $b_k \in B_k$ and $|x_k| = \delta^k_m$. Let $b_0 = \frac{m}{n} b_1 + \frac{m'}{n} b_2$, $x_0 = \frac{m}{n} x_1 + \frac{m'}{n} x_2$. From (5.1) we have $P_n(w_3) = b_0 + x_0$. From the definition of B_3, $b_0 \in B_3$. From the definition of x_0 we have that

$$\delta^3_n \leq |x_0| \leq \frac{m}{n}|x_1| + \frac{m'}{n}|x_2| = \frac{m}{n}\delta^1_m + \frac{m'}{n}\delta^2_{m'}. \tag{5.2}$$

If $m < N$, $\frac{m}{n}\delta^1_m$ is smaller than $\frac{m}{n}D$. Since $m < N$ and $n \geq ND/\epsilon$, we have $\frac{m}{n}\delta^1_m < \epsilon$. If $m \geq N$, we have $\delta^1_m < \epsilon$, so also $\frac{m}{n}\delta^1_m < \epsilon$. By the same reasoning we have that the second component is also smaller than ϵ. Therefore δ^3_n is smaller than 2ϵ for each $n \geq ND/\epsilon$, hence the sequence δ^3_n is indeed convergent do 0. Thus, all cluster points of $(P_n(w_3))$ must be in B_3. ∎

Lemma 5.3 *If A is a closed convex subset of C then $WF'(A)$ is weakly convex.*

Proof Let $v_1 = w_1 w_3 \ldots$ and $v_2 = w_2 w_4 \ldots$ be two words such that v_1, v_2, and w_i^ω are all in $WF'(A)$. We have to show that $v_3 = w_1 w_2 w_3 w_4 \ldots$ is also in $WF'(A)$. Let $x_n = P(w_1 w_2 w_3 \ldots w_n)$; (x_n) is a subsequence of $(P_n(v_3))$, so to show that $(P_n(v_3))$ has a cluster point in A, it is enough to show that (x_n) has a cluster point in A. However, each x_n is in A, since x_n a convex combination of $P(w_1), \ldots, P(w_n)$, and $P(w_i) = \lim P_n(w_i^\omega) \in A$. Since A is closed, (x_n) must have a cluster point in A. ∎

Lemma 5.4 *(a) If A is a non-trivial subset of C then $WF'(A)$ is not convex. (b) If A is not closed then $WF'(A)$ is not weakly convex.*

Proof Let $x \in A$. To show (b), let y_n be a sequence of elements of A convergent to $y \notin A$. To show (a), just take $y_n = y \notin A$.

Consider the infinite words u, v, w produced by the following (non-terminating) algorithm. Start with $u = x$, $v = x$, $w = xx$ (concatenation). For $n = 1, 2, \ldots$: Let l be the length of u. Append x^{nl} to u, y_n^{nnl} to v, $(xy_n^n)^{nl}$ to w. Let l be the length of v. Append x^{nl} to v, y_n^{nnl} to u, $(xy_n^n)^{nl}$ to w.

It can be easily seen that w is a shuffle of u and v. However, x is a cluster point of both u and v, but the only cluster point of w is y. Thus, $w \notin WF'(A)$, but $u, v \in WF'(A)$, so $WF'(A)$ is not convex. In case (b), we are shuffling only powers of x and y_n; their infinite repetitions $x^\omega, y_n^\omega \in WF'(A)$ ($\lim P_n(x^\omega) = P(x) = x \in A$), hence $WF'(A)$ is not even weakly convex. ■

Proof of Theorem 5.1 If A is trivial, obviously $WF'(A)$ is convex.

If A is not convex, let $x, y \in A$ such that $z = kx + (1-k)y \notin A$ for $k \in [0,1]$. Obviously, the infinite words x^ω and y^ω are in $WF(A)$ and $WF'(A)$, but we can shuffle them to obtain a word w such that $P_n(w)$ is convergent to z, thus $w \notin WF(A)$ and $w \notin WF'(A)$.

These two simple facts, together with the lemmas above, are enough to prove all items above. (Note that items 3, 4, 7, 8 are dual to items 1, 2, 5, 6.) ■

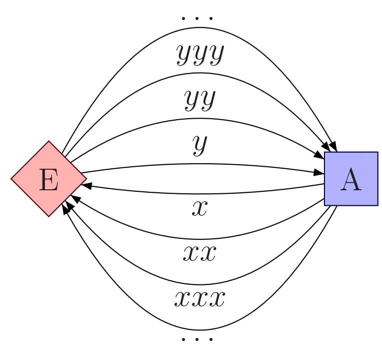
Note that in this theorem we assumed that each element of our space $[0,1]^d$ is allowed as a color of a move. The things may change if we restrict our color set C. For example, for $C = [0,1]$, $W = WF'([0,1] - \mathbb{Q})$ is not weakly convex from the theorem above. However, for $C = \{0,1\}$, $W \cap C^\omega$ is weakly convex, since there is no word w such that $w^\omega \in W$.

5.3 Positional Determinacy

By Theorems 5.1 and 4.7, if A is co-convex then $WF'(A)$ is concave and thus **finitely** half-positionally determined. However, the situation is different for infinite arenas.

Proposition 5.5 *If A is a non-trivial subset of C then $WF'(A)$ is not half-positionally determined.*

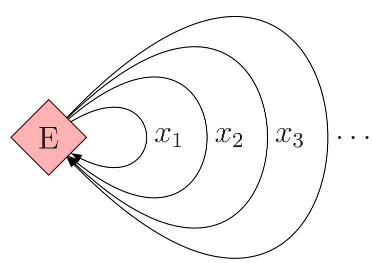
Proof of Proposition 5.5 Let $x \in C - A, y \in A$. Consider the game with two positions A and E where one can choose a move. A is Adam's position, E is Eve's position. In the position E Eve can choose a path going to A through k edges of color y, for each integer $k \geq 1$. Similarly, in A Adam can choose a path to E by k edges of color x, for all integers $k \geq 1$.



If Eve is using a positional strategy, always choosing the move generating the path y^k, Adam can win choosing x^{nk} in the n-th round. In this case the limit of $P_n(w)$ in x, hence Adam wins.

However, Eve can win by using a non-positional strategy. This strategy is to choose the move generating y^{nk} in the round n, where k is the number of x's generated in the last move of Adam. This ensures that y is a cluster point of $P_n(w)$, hence Eve wins. ∎

Proposition 5.6 *If A is not open then $WF(A)$ is not half-positionally determined.*



Proof of Proposition 5.6 Let $x = \lim_{n\to\infty} x_n$, where $x \in A$ and $x_n \notin A$. Consider the game with only one Eve's position E and moves E $\to$ E labeled x_n for each positive integer n. Eve has only non-positional winning strategies here. ∎

5.4 Simple Open Set

In this section we show that $WF(A)$ is half-positional for very simple closed sets A. The problem remains unsolved for more complicated sets.

Theorem 5.7 *Let $C = [0, 1]$, $A = [0, 1/2)$. The condition*

$$WF(A) = \{w : \limsup P_n(w) < 1/2\}$$

is half-positional.

Proof of Theorem 5.7

Let $G = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$ be an arena. Consider the following prefix dependent winning condition for $x \in [0, 1]$:

$$WL_x = \{w : \forall_n P_n(w) \leq x\} \tag{5.3}$$

Let $L_x = \mathrm{Win}_E(G, WL_x)$, i. e. the set of positions v such that there exists a winning strategy for Eve in the game starting from the position v. We will use the following lemma:

Lemma 5.8 *Let $x < 1/2$. In L_x Eve has a positional winning strategy in $(G, WF(A))$.*

To apply Lemma 3.5 it remains to prove that if L_x is empty for each $x < 1/2$ then Adam has a winning strategy everywhere. Let (a_n) be an increasing sequence convergent to $1/2$. The strategy is as follows:

- For each $i = 1, 2, \dots$:
  - Let t be the current time (i.e., length of the play so far), and v be the current position. Since $v \notin L_{a_i}$, we know that Adam has a strategy which guarantees that after some time t' we get $P(w) > a_i$, where w is the color word obtained from time t to t'. Adam uses this strategy until this happens.

If Adam uses this strategy, we get an infinite play whose color word is $w = w_1 w_2 w_3 \dots$, where $P(w_i) > a_i$. One can easily check that, for each i, there will be a t such that $P_t(w) > a_i$. Thus, $\limsup P_n(w)$ is at least $1/2$. ■

Proof of Lemma 5.8

Let (G^x, WP_1) (WP_1 is the parity condition over $C = \{0, 1\}$) be the game where:

- $\mathrm{Pos}^x_X = \mathrm{Pos}_X \times \mathbb{R}$ for $X \in \{A, E\}$,
- For each move $v \xrightarrow{t} w \in \mathrm{Mov}$ and $z \geq 0$ we have a move $(v, z) \xrightarrow{0} (w, z + x - t)$ in Mov^x,

- For each move $v \xrightarrow{t} w \in \text{Mov}$ and $z < 0$ we have a move $(v, z) \xrightarrow{1} (w, z)$ in Mov^x.

The number z in position $(v, z) \in \text{Pos}^x$ defines Eve's reserve. Eve wins all infinite plays where this reserve does not fall beyond 0 (if z falls beyond 0, then it stays there).

The plays in (G^x, WP_1) can be projected to (G, WL_x). And vice versa, a play in (G, WL_x) starting in v can be raised to a play in (G^x, WP_1). One can easily show that projecting and raising plays preserves the winner, provided that in (G^x, WP_1) we start in $(v, 0)$ for some v. Hence $L_x = \{v : (v, 0) \in \text{Win}_E(G^x, WP_1)\}$.

The parity condition WP_1 is positionally determined, thus the game (G^x, WP_1) we constructed is positionally determined. Let s' be a positional strategy winning in $\text{Win}_E(G^x, WP_1)$. Clearly if $z_1 \le z_2$ then $(v, z_1) \in \text{Win}_E(G^x, WP_1)$ implies $(v, z_2) \in \text{Win}_E(G^x, WP_1)$. Let M be the set of v such that $(v, z) \in \text{Win}_E(G^x, WP_1)$ for some $z \ge 0$; we have $L_x \subseteq M$. Let $x < y < 1/2$. Consider the following strategy in M:

$$s(v) = \pi(s'(v, z(v) + (y - x))) \tag{5.4}$$

where π is the projection from Mov^x to Mov, and

$$z(v) = \inf\{z : \ (v, z) \in \text{Win}_E(G^x)\}. \tag{5.5}$$

Let (G^y, WP_1) be a game constructed analogically to (G^x, WP_1). One can easily check that each game starting in $v \in M$ which is consistent with s projects to some play in (G^y, WP_1) winning for Eve and starting in $(v, z(v))$. Hence the play in $(G, WF(A))$ satisfies the winning condition $WF(A)$. ■

This theorem can be generalized to the following:

Corollary 5.9 *Let $A = f^{-1}(\{x \in \mathbb{R} : x < 0\})$ for some affine function $f : C \to \mathbb{R}$. Then, the condition $WF(A)$ is half-positional.*

Proof Let $a_0 = \min f(C), a_1 = \max f(C)$. Let h be such that $0 \le 1/2 + ha_0 \le 1/2 + ha_1 \le 1$. Let G' be the arena like G, except that we replace each color c with $t(c) = 1/2 + hf(c)$. By our assumption, G' is an arena over $[0, 1]$, and one can easily check that Eve wins a play in $(G, WF(A))$ iff she wins the corresponding play in $(G', WF([0, 1/2)))$. ■

5.5 Summary

The following table summarizes what we know about concavity and half-positional determinacy of geometrical conditions. In every point except No. 0 we assume that A is non-trivial, i.e. $\emptyset \neq A \neq C$. The first two columns specify assumptions about A and whether we consider $WF(A)$ or $WF'(A)$, and the last three answer whether the considered condition is concave and whether it has finite and/or infinite half-positional determinacy. Negative answer means that the answer is negative for all sets A in the given class; the question mark means that the given problem has not been solved yet (but we suppose that the answer is positive).

No.	A	condition	concavity	finite	infinite
0	trivial	$WF'(A)$ or $WF(A)$	yes	yes	yes
1	not co-convex	$WF'(A)$ or $WF(A)$	no	no	no
2	co-convex	$WF'(A)$	yes	yes	no
3	co-convex, not open	$WF(A)$	no	yes?	no
4	co-convex, open	$WF(A)$	weak only	yes?	yes?
5	$[\frac{1}{2}, 1] \subset [0, 1]$	$WF(A)$	weak only	yes	yes

Note that, for any set A which is co-convex and non-trivial, $WF'(A)$ is finitely half-positionally determined, but not infinitely half-positionally determined. This shows a big difference between half-positional determinacy on finite and infinite arenas.

Chapter 6

Games and Finite Automata

Infinite games are strongly linked to automata theory. An accepting run of an alternating automaton (on a given tree) can be presented as a winning strategy in a certain game between two players. Parity games are related to automata on infinite structures with parity acceptance condition. For example, positional determinacy of parity games is used in modern proofs of Rabin's complementation theorem for finite automata on infinite trees with Müller (or, equivalently, parity) acceptance condition. See [GTW02] for more links between infinite games, automata, and logic.

In this chapter we concentrate on the links between our subject and finite automata. Winning conditions are languages of infinite words over C, and many of those which are used in theory and practice are ω-regular. Examples include parity conditions, Rabin conditions (unions of parity conditions), and Müller conditions (which are defined in the terms of colors which appear infinitely often). There are many equivalent definitions of the class of ω-regular languages, which generalizes the class of regular languages of finite words. We will use deterministic finite automata with parity acceptance condition — a language $L \subseteq C^\omega$ is **ω-regular** if it is accepted by an automaton of this kind. Other definitions use ω-regular expressions (which are a very effective method of expressing ω-regular languages, and are used in many places in this thesis), other kinds of automata (e.g. nondeterministic Büchi automata), or notions of logic. It is a well known fact that the class of ω-regular languages is closed under operations such as union, intersection, negation, and homomorphic preimages and images. Since finite automata provide nice finite descriptions for ω-regular languages, it is possible to give algorithms which check properties of an ω-regular winning condition, given the automaton that accepts it.

First, we present the definition of a DFA with parity acceptance condition. In the next section we show a class of half-positional winning conditions defined using a finite automaton (on finite words). In the next two sections we show what can be said about finite half-positional determinacy of a winning condition which is ω-regular. Precisely, we show that if an ω-regular winning condition is not half-positional then this is witnessed by a very simple arena, which will lead us to an algorithm which decides whether given winning condition is finitely half-positional. In the last section we show that concavity of an ω-regular language is also decidable.

6.1 Definitions and Prefix Independence

We start by defining a DFA with parity acceptance condition.

Definition 6.1 *A* **deterministic finite automaton (DFA) on infinite words with parity acceptance condition** *is a tuple* $A = (Q, q_I, \delta, \mathrm{rank})$, *where* Q *is a finite set of states,* $q_I \in Q$ *the initial state,* $\mathrm{rank} : Q \to \{0, \dots, d\}$, *and* $\delta : Q \times C \to Q$. *We extend the definition of* δ *to* $\delta : Q \times C^* \to Q$ *by* $\delta(q, \epsilon) = q, \delta(q, wu) = \delta(\delta(q, w), u)$ *for* $w \in C^*, u \in C$. *For* $w \in C^\omega$, *let* $q_0(w) = q_I$ *and* $q_{n+1}(w) = \delta(q_n, w_{n+1}) = \delta(q_I, w_0 \dots w_{n+1})$. *We say that the word* $w \in C^\omega$ *is* **accepted** *by* A *iff* $\limsup_{n\to\infty} \mathrm{rank}(q_n(w))$ *is even. The set of all words accepted by* A *is called* **language accepted by** A *(or,* **recognized by** A*) and denoted* L_A.

Since we are speaking about ω-regular winning conditions which are prefix independent, we can assume that our automaton has additional properties — strong connectedness and irrelevance of initial state.

Proposition 6.2 *Let* $A = (Q, q_I, \delta, \mathrm{rank})$.

(a) If $A' = (Q, q'_I, \delta, \mathrm{rank})$ *where* $q'_I = \delta(q_I, u)$ *for* $u \in L^*$, *then* $w \in L_{A'}$ *iff* $uw \in L_A$.

(b) If $L_{(Q',q,\delta,\mathrm{rank})}$ *does not depend on* $q \in Q$ *then* L_A *is prefix independent.*

(c) If A *is strongly connected (i.e. for each* $q, q' \in Q'$ *there is a word* $w \in C^*$ *such that* $\delta(q, w) = q'$*), and* L_A *is prefix independent, then* $L_{(Q,q,\delta,\mathrm{rank})}$ *does not depend on* $q \in Q$.

(d) If L_A *is prefix independent, then there is a subset* $Q' \subseteq Q$ *and* $q'_I \in Q'$ *such that* $L_A = L_{A'}$ *for* $A' = (Q', q'_I, \delta, \mathrm{rank})$, *and* A' *is strongly connected,.*

Proof (a) $q_{m+n}(uw) = \delta(q_I, w_{|m+n}) = \delta(\delta(q_I, u), w_{|n}) = q'_n(w)$.

(b) Let $u \in C^*$ be a word of length m, and $w \in C^\omega$. From (a) we easily get that $w \in L_A$ iff $uw \in L_A$.

(c) Let $A' = (Q, q', \delta, \mathrm{rank})$. Let w_0 be a word such that $q' = \delta(q_I, w_0)$. We have $w \in L_{A'}$ iff $w_0 w \in L_A$, which is equivalent to $w \in L_A$.

(d) Like in the proof of (c) we can change the initial state. If some states are not reachable from the current initial state, we can remove them from out automaton. Repeat until the obtained automaton is strongly connected. ■

Strong connectedness is not sufficient for prefix independence — for example, the language $(\mathbf{b}^*\mathbf{a}\mathbf{b}^*\mathbf{a})^*\mathbf{b}^\omega \subseteq \{\mathbf{a}, \mathbf{b}\}^\omega$ is not prefix independent, but it can be recognized with a strongly connected automaton with 2 states. Prefix independence of an ω-regular language can be checked using standard techniques from automata theory (building automata recognizing $L_A - L_{A'}$ and $L_{A'} - L_A$, for each automaton A' with changed initial state, and testing its emptiness).

6.2 Monotonic Automata

In this section we show yet another class of half-positionally determined winning conditions which is based on an idea coming from automata theory, and guarantees half-positional determinacy even for infinite arenas. We need to introduce a special kind of deterministic finite automaton (on finite words).

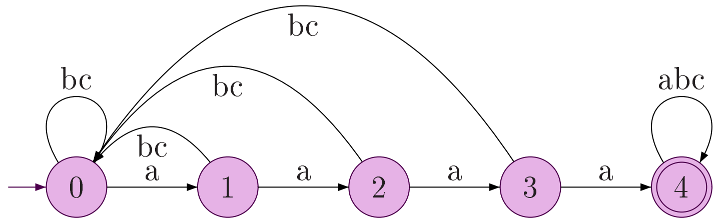
Definition 6.3 *A* **monotonic automaton** $A = (n, \delta)$ *over an alphabet* C *is a deterministic finite automaton (on finite words) where:*

- *the set of states is* $Q = \{0, \dots, n\}$*;*
- *the initial state is 0, and the accepting state is* n*;*
- *the transition function* $\delta : Q \times C \to Q$ *is monotonic in the first component, i.e., if* $q \leq q'$ *then* $\delta(q, c) \leq \delta(q', c)$.

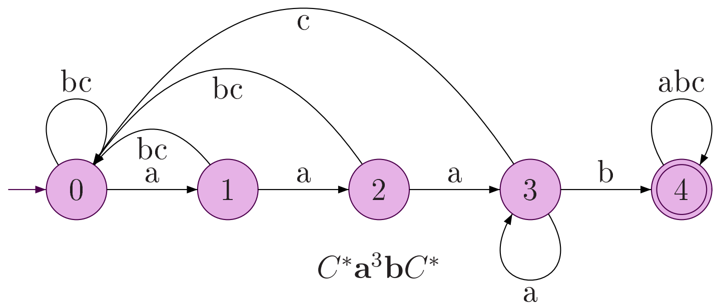
Actually, we need not require that the set of states is finite. All the results presented here except for Theorem 7.13 and the remark about finite memory of Adam can be proven with a weaker assumption that Q has a minimum (initial state) and its each non-empty subset has a maximum.

The function δ is extended to C^* as in Definition 6.1; this extension is still monotonic. By L_A we denote *the language accepted (recognized) by* A, i.e., the set of words $w \in C^*$ such that $\delta(0, w) = n$.

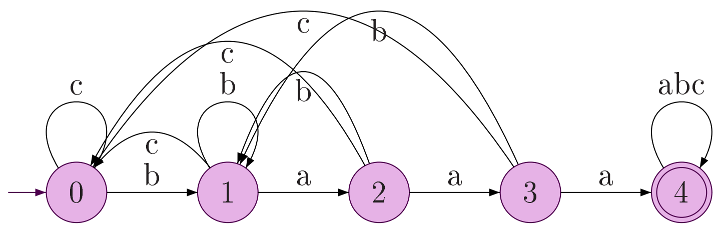
Example 6.4 *Let* $C = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$*. Monotonic automata can recognize the following languages:* $C^*\mathbf{a}^nC^*$*,* $C^*\mathbf{a}^{n-1}\mathbf{b}C^*$*,* $C^*\mathbf{b}\mathbf{a}^{n-1}C^*$*. Monotonic automata cannot recognize the following languages:* $C^*\mathbf{a}^2\mathbf{b}^2C^*$*,* $C^*\mathbf{bab}C^*$*,* $C^*\mathbf{bac}C^*$*.*



$C^*\mathbf{a}^4C^*$



$C^*\mathbf{a}^3\mathbf{b}C^*$



$C^*\mathbf{ba}^3C^*$

The pictures illustrate automata recognizing these languages, for $n = 4$. (To show that the other languages are not recognizable by monotonic automata, one can use e.g. Theorem 6.6 or Proposition 6.7 below.)

Definition 6.5 *A* **monotonic condition** *is a winning condition of the form $WM_A = C^\omega - L_A^\omega$ for some monotonic automaton A.*

Note that if $w \in L_A$ then $uw \in L_A$ for each $u \in C^*$. Hence $L_A = C^*L_A$, thus L_A and WM_A are prefix independent for each A. Also, L_A^ω is equal to $L_A(C^*L_A)^\omega = (L_AC^*)^\omega$, hence without affecting WM_A we can assume that $\delta(n, c) = n$ for each c.

Theorem 6.6 *Any monotonic condition is half-positional.*

In Section 8.2 we analyze memory required by Adam to win in his winning set.

Proof of Theorem 6.6

Let $A = (n, \delta)$ be a monotonic automaton, and $G = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$ be an arena. We will show that the game (G, WM_A) is half-positionally determined.

We will construct a new game on the arena $G' = (\mathrm{Pos}'_A, \mathrm{Pos}'_E, \mathrm{Mov}')$ over the set of colors $C' = \{0, 1\}$ with the parity condition WP_1, where $\mathrm{Pos}'_X = \mathrm{Pos}_X \times Q$. For each move $v_1 \xrightarrow{c} v_2$ in G, in G' we have moves $(v_1, q) \xrightarrow{c'} (v_2, \delta(q, c))$ for each $q \leq n$; $c' = 0$ for $q < n$ and 1 for $q = n$. (We assumed that $\delta(n, c) = n$, which means that after reaching $q = n$ Adam will win, unless the play ends finitely.)

The game (G', WP_1) is positionally determined, therefore Pos' can be split into the winning sets of both players, Win'_A and Win'_E, and in Win'_E we have a positional winning strategy for Eve, $s' : \mathrm{Win}'_E \to \mathrm{Mov}$. Let $M \subseteq \mathrm{Pos}$ be the set of v such that $(v, 0) \in \mathrm{Win}'_E$.

There are two cases:

1. $M = \emptyset$. We will show that Adam has a winning strategy in (G, WM_A) from each position. This strategy is implemented by the following algorithm:

    - Let v_1 be the starting position (after $R_1 = 0$ moves);
    - For $i = 1, 2, 3, \ldots$:
    - After R_i moves we are in position v_i. In G', Adam has a strategy ensuring reaching from $(v_i, 0)$ to some state belonging to the set $N = \{(v, n) : v \in G\}$. Adam uses a projection of this strategy (ignoring R_i moves which have been made before reaching v_i), until he reaches N in G'. Let v_{i+1} be the vertex reached in G.

      The word w created by colors of moves made in meantime satisfies $\delta(0, w) = n$.

    The word created between the R_i-th and R_{i+1}-th move belongs to L_A, therefore the infinite word created during the whole play does not belong to WM_A.

2. $M \neq \emptyset$. We will show that Eve has a positional winning strategy in (G, WM_A) for each starting position $v_0 \in M$.

    Note that if $q_1 < q_2$ and $(v, q_2) \in \mathrm{Win}'_E$ then also $(v, q_1) \in \mathrm{Win}'_E$. (The situation with smaller q is better for Eve.) For $v \in M$ we denote by $H(v)$ the greatest q for each $(v, q) \in \mathrm{Win}'_E$.

We define Eve's positional strategy in the game (G, WM_A) in the set M in the following way: for $v \in M$, $s(v) = \alpha(s'(v, H(v)))$, where $\alpha(p')$ for a move $p' \in \mathrm{Mov}'$ is the move in Mov such that p' is derived from p (in case if there are many such moves, $\alpha(p')$ can be any one of them).

Let π be a play consistent with the strategy s, and $v_i = \mathrm{target}(\pi_i)$ for $i > 0$. Let $q_i = \delta(0, v_1 v_2 \dots v_i)$ be the state of the automaton when reaching v_i. We will show by induction that for all i we have $q_i \leq H(v_i)$, and therefore $q_i < n$ and $v_i \in M$. Obviously $q_0 = 0 \leq H(v_0)$. Now, assume that $q_i \leq H(v_i)$; we will show that $q_{i+1} \leq H(v_{i+1})$.

Suppose $v_i \in \mathrm{Pos}_E$. This means that $v_{i+1} = \mathrm{target}(s(v_i))$, and thus, $\mathrm{target}(s'(v_i, H(v_i)))$ is (v_{i+1}, q) for some q. Since $q_i \leq H(v_i)$, $q_{i+1} = \delta(v_{i+1}, \mathrm{rank}(s(v_i)))$, $q = \delta(H(v_i), \mathrm{rank}(s(v_i)))$, and δ is monotonic, we have $q_{i+1} \leq q$. On the other hand, we know that $(v_{i+1}, q) \in Win'_E$, therefore $q \leq H(v_{i+1})$. Hence indeed $q_{i+1} \leq H(v_{i+1})$.

Now, suppose $v_i \in \mathrm{Pos}_A$. Then $v_{i+1} = \mathrm{target}(p)$ for some move p from v_i. The move p gives rise to moves $p_1 = ((v_i, q_i) \xrightarrow{0} (v_{i+1}, q_{i+1}))$ and $p_2 = ((v_i, H(v_i)) \xrightarrow{0} (v_{i+1}, q))$ in Mov'. Since $q_i \leq H(v_i)$, by monotonicity of δ we obtain $q_{i+1} \leq q$. We also have $q \leq H(v_{i+1})$, since otherwise Adam could leave Eve's winning set in G' (using the move p_2).

Since for each i we have $q_i \leq H(v_i) < n$, the word $v_1 v_2 \dots$ has to belong to WM_A.

Half-positional determinacy follows from Lemma 3.5.

■

From this theorem, together with Example 6.4 above, one can see that e.g. WA_n, the complement of the set of words containing a^n infinitely many times, is monotonic, and thus half-positionally determined.

For $n = 1$ the set WA_n is just a co-Büchi condition. However, for $n > 1$ it is easily shown that WA_n is not (fully) positionally determined, and also that it is not concave. For example, for $n = 2$ the word $(\mathbf{bababbabab})^\omega$ is a shuffle of $(\mathbf{bbbaa})^\omega$ and $(\mathbf{aabbb})^\omega$.

Proposition 6.7 *All monotonic conditions are weakly concave.*

Proof Let $A = (n, \delta)$ be a monotonic automaton. We will show a stronger property, namely that, for each sequence of words $w_1, w_2, \dots$, if $\forall_i w_i^\omega \in L_A^\omega$, then $w_1 w_2 w_3 \dots \in L_A^\omega$. (We don't use the assumption that $w_1 w_3 w_5 \dots \in L_A^\omega$ and $w_2 w_4 w_6 \in L_A^\omega$.)

We will assume that $\delta(n, c) = n$ for each c.

Since $w_i^\omega \notin WM_A$, we have that $\delta(q, w_i) > q$ for each $q < n$. Otherwise, if for some q we had $\delta(q, w_i) \leq q$, then, from monotonicity of δ, $\delta(q', w_i) \leq q$ for each $q' \leq q$, thus A will not accept any prefix of w_i^ω, because we will never reach the state n starting from the state $0 \leq q$.

Hence $\delta(0, w_{m+1}w_{m+2}w_{m+3} \dots w_{m+n}) = n$ for each $m \in \mathbb{N}$. Thus, the word $w_1w_2w_3\dots$ is indeed in L_A^ω, and is not in WM_A. ■

Proposition 6.8 *Monotonic conditions are closed under finite union.*

Proof It can be easily shown that $C^\omega - W_{A_1} \cup W_{A_2} = L_{A_1}^\omega \cap L_{A_2}^\omega = (C^*L_{A_1})^\omega \cap (C^*L_{A_2})^\omega = (C^*L_{A_1}C^*L_{A_2})^\omega = (L_{A_1}L_{A_2})^\omega$. The language $L_{A_1}L_{A_2}$ is recognized by the monotonic automaton $A_s = (n_1 + n_2, \delta)$, where $\delta(q, c) = \delta_1(q, c)$ for $0 \leq q < n_1$ and $\delta(n_1 + q, c) = n_1 + \delta_2(q, c)$ for $0 \leq q \leq n_2$. ■

Monotonic conditions are not closed under other Boolean operations.

6.3 Simplifying the Witness Arena

To show that finite half-positional determinacy of winning conditions which are prefix independent ω-regular languages is decidable, we first need to show that if W is not finitely half-positional, then it is witnessed by a simple arena.

Theorem 6.9 *Let W be a winning condition accepted by a deterministic finite automaton with parity acceptance condition*

$$A = (Q, q_I, \delta, \mathrm{rank} : Q \to \{0 \dots d\})$$

(see Definition 6.1). If W is not finitely half-positional then there is a witness arena (i.e. such that Eve has a winning strategy, but no positional winning strategy) where there is only one Eve's position, and only two moves from this position. (There is no restriction on Adam's moves and positions.)

Proof Let G be any finite witness arena. Without loss of generality we can assume that Eve has a winning strategy everywhere (otherwise we restrict our arena to Eve's winning set). First, we will show how to reduce the number of Eve's positions to just one. Then, we will show how to remove unnecessary moves.

Let $G^0 = (\mathrm{Pos}_A \times Q, \mathrm{Pos}_E \times Q, \mathrm{Mov}^0)$ and $G^1 = (\mathrm{Pos}_A \times Q, \mathrm{Pos}_E \times Q, \mathrm{Mov}^1)$ where for each move $v_1 \xrightarrow{c} v_2$ in G and each state q we have corresponding moves $(v_1, q) \xrightarrow{c} (v_2, \delta(q, c))$ in Mov^0 and $(v_1, q) \xrightarrow{\mathrm{rank}(q)} (v_2, \delta(q, c))$

in Mov^1. The three games (G, W), (G^0, W) and (G^1, WP_d) are equivalent: each play in one of them can be interpreted as a play in each another, and the winner does not change for infinite plays.

More specifically, the correspondence between G^0 and G is based on replacing each position v by a set $U(v) = \{(v, q) : q \in Q\}$. For each element w of $U(v)$ if there is a move $v \xrightarrow{c} v'$, then there exists a move $w \xrightarrow{c} w'$ for some $w' \in U(v')$. Also, if there is a move $w \xrightarrow{c} w'$ for some $w \in U(v), w' \in U(v')$, then there is a move $v \xrightarrow{c} v'$.

Between G^0 and G^1, the only difference is the rank function, thus plays in one of them can be interpreted as plays in the other in the obvious way. (There is a slight technical difficulty, as it is possible that several moves in G^0 correspond to the same move in G^1. Then, we can pick any of them when interpreting a play in G^1 as a play in G^0.) To see that the winner does not change, take a play $\pi = \pi_0\pi_1\pi_2\ldots$ in G^0, and let $(v_i, q_i) = \mathrm{source}(\pi_i)$. In the game G^0 the color of π_i is c_i, and the color of the corresponding move in G^1 is $\mathrm{rank}(q_i)$. From definition of G^0, by induction we have $q_i = \delta(q_I, c_1c_2\ldots c_i)$, so Eve wins iff $\limsup \mathrm{rank}(q_i)$ is even — which agrees with the parity condition in G^1.

Since Eve has a winning strategy in (G, W), she also has a winning strategy in (G^1, WP_d). This game is positionally determined, so she also has a positional strategy here. She can use the corresponding positional strategy in (G^0, W) too.

Let s be Eve's positional winning strategy in G^0. Let

$$N(s) = \{v : \exists q_1 \exists q_2\ \pi_1(\mathrm{target}(s(v, q_1))) \neq \pi_1(\mathrm{target}(s(v, q_2)))\},$$

i.e. the set of positions where s is not positional as a strategy in G. Since the arena is finite, we can assume without loss of generality that there is no positional winning strategy s' in G^0 such that $N(s') \subsetneq N(s)$.

If $N(s)$ was empty, then we could use s as a positional strategy in G, which would contradict our assumption that G is a witness arena. Let $v_0 \in N(s)$. We construct a new arena G^2 from G^0 in two steps.

First, merge $\{v_0\} \times Q$ into a single position v_0. Eve can transform s into a winning (non-positional) strategy s_1 in this new game — the only difference is that in v_0 she needs to remember in what state q she is currently, and move according to $s(v_0, q)$.

Then, for all Eve's positions except v_0, remove all moves which are not used by s (and thus by s_1). Eve still wins by s_1, since she did not lose any options used by s_1. Now, transfer all Eve's positions except v_0 to Adam. Eve still wins by s_1, since there was no choice in these positions.

Thus, we obtained an arena G^2 with only one Eve's position v_0, where she has a winning strategy from v_0.

Eve has no winning positional strategy in G^2. Indeed, suppose that such a strategy exists. Then it can be simulated without changing the winner (in the natural way) by a strategy s_2 in G, positional in all positions except $N(s) - \{v_0\}$. Let G_* be G without moves which are not used by s_2 — s_2 remains a winning strategy on G_*. Let G^0_* be the arena obtained from G_* in the same way as we obtained G^0 from G. Let s_3 be Eve's positional winning strategy on G^0_* (which exists since Eve had a winning strategy on G_*); as a strategy on G^0, it is also winning, and has $N(s_3) \subsetneq N(s)$. This contradicts our assumption that $N(s)$ is minimal, so Eve has no winning positional strategy in G^2,

Hence we have found a witness arena where $|Pos_E| = 1$. (Note that we can assume that Eve has at most $|Q|$ moves here — Eve's positional winning strategy on G_0 cannot use more than $|Q|$ moves from positions derived from v_0, so unused moves can be safely removed.)

Now, suppose that G is an arbitrary witness arena with only one Eve's position. We will construct a new arena with only two possible moves for Eve. The construction goes as follows:

- We construct G^0 as before.
- We start with $G^3 = G^0$. Let s be Eve's winning strategy in G^3.
- For each of Eve's $|Q|$ positions in G^3, we remove all moves except the one which is used by s.
- ($\star$) Let v_1 and v_2 be two Eve's positions in G^3.
- We merge Eve's positions v_1 and v_2 into one, v_0.
- Eve still has a winning strategy everywhere in this new game (by a reasoning similar to one we used for G^2). We check if Eve has a positional winning strategy.
- If yes, we remove the move which is not used in v_0, and go back to ($\star$). (Two distinct Eve's positions in G^3 must still exist — if we were able to merge all Eve's positions into one, it would mean that G was positionally determined.)
- Otherwise G^3 is now a witness arena. In all Eve's positions except v_0 there is only one move, so we can safely transfer them to Adam, and G^3 will remain a witness arena.
- In G^3 we have now only one Eve's position (v_0) and only two Eve's moves — one inherited from v_1 and one inherited from v_2.

∎

6.4 Decidability

Theorem 6.10 *Let W be a (prefix independent) ω-regular winning condition recognized by a DFA with parity acceptance condition*

$$A = (Q, q_I, \delta, \mathrm{rank} : Q \to \{0 \dots d\})$$

with n states. Then finite half-positional determinacy of W is decidable in time $n^{O(n^2)}$.

Proof It is enough to check all possible witness arenas which agree with the hypothesis of Theorem 6.9. Such arena consists of (the only) Eve's position E from which she can move to A_1 by move p_1 or to A_2 by move p_2. Since we are working on ϵ-arenas (see Section 2.5), we can assume that $A_i \neq E$, and also that these two moves are ϵ-moves; otherwise we add a new Adam's position "in the middle of the move" and connect it with an ϵ-move. Adam has a choice of word w by which he will return to E from A_i. (In general it is possible that Adam can choose to never return to E. However, if such infinite path was winning for Eve, he would not choose it, and if it would be winning for Adam, Eve would never hope to win by choosing to move to A_i, thus she would always have to choose the other move, and thus our arena wouldn't be a witness.) Let L_i be the set of all possible Adam's return words from A_i to E.

Let $T(w) : Q \to \{0, \dots, d\} \times Q$ be the function defined as follows: $T(w)(q) = (r, q')$ iff $\delta(q, w) = q'$ and the greatest rank visited during these transitions is r. The function $T(w)$ contains all the information about $w \in L_i$ which is important for our game: if $T(w_1) = T(w_2)$ then it does not matter whether Adam chooses to return by w_1 or w_2 (the winner does not change). Thus, instead of Adam choosing a word w from L_i, we can assume that Adam chooses a function t from $T(L_i) \subseteq T(C^*) \subseteq (Q \times \{0, \dots, d\})^Q$.

For non-empty $R \subseteq \{0, \dots, d\}$, let $\mathrm{best}^{\mathrm{A}}(R)$ be the priority which is the best for Adam, i.e. the greatest odd element of R, or the smallest even one if there are no odd priorities in R. We also put $\mathrm{best}^{\mathrm{A}}(\emptyset) = \bot$.

For $T \subseteq (Q \times \{0, \dots, d\})^Q$, let

$$U(T)(q_1, q_2) = \mathrm{best}^{\mathrm{A}}(\{d : \exists t \in T \; t(q_1) = (d, q_2)\}).$$

Again, the function $U_i = U(T(L_i)) : Q \times Q \to \{\bot, 0, \dots, d\}$ contains all the information about L_i which is important for our game — if Adam can go from q_1 to q_2 by one of two words w_1 and w_2 having the highest priorities d_1 or d_2, respectively, he will never want to choose the one which is worse to him.

Our algorithm checks all possible functions U_i. For this, we need to know whether a particular function $U : Q \times Q \to \{\bot, 0, \ldots, d\}$ is of form $U(T(L_i))$ for some L_i. This can be done in following way. We start with $V(q, q) = \bot$. Generate all elements of $T(L_i)$. This can be done by doing a search (e.g. breadth first search) on the graph whose vertices are $T(w)$ and edges are $T(w) \to T(wc)$ ($T(wc)$ obviously depends only on $T(w)$). For each of these elements, we check if it does not give Adam a better option than U is supposed to give — i.e. for some q_1 we have $T(wc)(q_1) = (q_2, d)$ and $d = \text{best}^{\text{A}}(d, U(q_1, q_2))$. If it does not, we add $T(w)$ to our set T and update V: for each q_1, $T(wc)(q_1) = (q_2, d)$, we set $V(q_1, q_2) := \text{best}^{\text{A}}(d, V(q_1, q_2))$. If after checking all elements of $T(L_i)$ we get $V = U$, then $U = U(T)$. Otherwise, there is no L such that $U = U(T(L))$.

The general algorithm is as follows:

- Generate all possible functions U of form $U(T(L))$.
- For each possible function U_1 describing Adam's possible moves after Eve's move p_1 such that Eve cannot win by always moving with p_1:
- For each U_2 (likewise):
- Check if Eve can win by using a non-positional strategy. (This is done easily by constructing an equivalent parity game which has $3|Q|$ vertices: $\{E, A_1, A_2\} \times Q$.) If yes, then we found a witness arena.

Time complexity of the first step is $O(d^{O(|Q|^2)}(d|Q|)^{|Q|}|C|)$ (for each of $d^{O(|Q|^2)}$ functions, we have to do a BFS on a graph of size $(d|Q|)^{|Q|}$). The parity game in the fourth step can be solved with one of the known algorithm for solving parity games, e.g. with the classical one in time $O(O(|Q|)^{d/2})$. This is done $O(d^{O(|Q|^2)})$ times. Thus, the whole algorithm runs in time $O(d^{O(|Q|^2)}|Q|^{|Q|}|C|)$. ∎

In the proof above the witness arena we find is an ϵ-arena: we did not assign any colors to moves p_1 and p_2. If we want to check whether the given condition is A-half-positional or B-half-positional (see Section 2.5), similar constructions work. For B-half-positional determinacy, we need to not only choose the sets U_1 and U_2, but also choose specific colors c_1 and c_2 for both moves p_1 and p_2 in the algorithm above, and take care of the case when $A_1 = E$ or $A_2 = E$. For A-half-positional determinacy, we need to choose specific colors for targets of these two moves, and also a color for Eve's position E.

Once we know that an ω-regular winning condition W is indeed finitely half-positional, we can use the following algorithm to solve a game.

Proposition 6.11 *Suppose that G is an arena with n positions, and W is finitely half-positional and ω-regular, given by a DFA with parity acceptance condition on infinite words using s states and d ranks.*

Then the winning sets for Eve and Adam in the game (G, W) can be found in time $O((ns)^{d/2})$, and Eve's positional strategy can be found in time $O((ns)^{d/2}t)$, where $t = \sum_{v \in \mathrm{Pos}_E} \log |v\mathrm{Mov}|$, where $|v\mathrm{Mov}|$ is the number of moves outgoing from v.

Proof As in the proof of Theorem 6.9, we transform our game (G, W) (with n positions) into a parity game (G^2, WP_d) (with ns positions). Winning sets and positional strategies in such a game can be determined in time $O((ns)^{d/2})$ (see e.g. [GTW02]).

To obtain Eve's strategy, we use the following reduction of the problem of finding Eve's positional winning strategy to the problem of finding the winning sets for both players (which actually works for all finitely half-positional winning conditions — not only ω-regular ones). If we remove Eve's move which is not used by her winning strategy, Win_E does not change. Thus, we can try to remove half of moves outgoing from one of Eve's positions, and see if Win_E changes — if yes, then Eve should use one of removed moves, otherwise Eve should use one of the remaining moves. We continue doing this until only one move remains in each Eve's position. ■

6.5 ω-regular Concave Conditions

The following proposition shows that concavity (see Chapter 9) is decidable for ω-regular language in polynomial time. As shown in Theorem 4.7, concave winning conditions are finitely half-positional.

Proposition 6.12 *Suppose that a winning condition W is given by a DFA with parity acceptance condition using s states and d ranks. Then there exists a $O(s^6 d^3 |C|)$ algorithm determining whether W is concave (or convex).*

Definition 6.13 *For $q_1, q_2, q_3, r_1, r_2, r_3 \in Q$, $n_1, n_2, n_3 \in \{\bot, 0, \ldots, d\}$, we say that $P(q_1, r_1, n_1,\ q_2, r_2, n_2,\ q_3, r_3, n_3)$ iff there exists a word $w_3 \in C^*$ being a shuffle of w_1 and w_2 such that for each $k \in \{1, 2, 3\}$ we have $\delta(q_k, w_k) = r_k$,* *and n_k is the greatest rank of states appearing while the automaton works on w_k starting from q_k, i.e. $n_k = \max_{w_k=uv} \operatorname{rank}(\delta(q_k, u))$. In case if $w_k = \epsilon$ we take $n_k = \perp$.*

Lemma 6.14 *L_A is not convex iff for some $q_1, q_2, q_3, m_1, m_2, m_3, n_1, n_2, n_3$ we have $P(q_I, q_1, m_1,\ q_I, q_2, m_2,\ q_I, q_3, m_3)$ and $P(q_1, q_1, n_1,\ q_2, q_2, n_2,\ q_3, q_3, n_3)$ and n_1, n_2 are even and n_3 is odd. ($\perp$ is considered neither even nor odd.)*

Proof

($\leftarrow$) Let u_1, u_2 and u_3 be the words from 6.13 which are witnesses for $P(q_I, q_1, m_1,\ q_I, q_2, m_2,\ q_I, q_3, m_3)$, and v_1, v_2 and v_3 be the words which are witnesses for $P(q_1, q_1, n_1,\ q_2, q_2, n_2,\ q_3, q_3, n_3)$. Let $w_k = u_k v_k^\omega$. It can be easily shown that w_3 is a shuffle of w_1 and w_2 and $w_1, w_2 \in L_A$ but $w_3 \notin L_A$.

($\rightarrow$) Suppose that L_A is not convex, i.e., w^3 is a shuffle of w^1 and w^2, and $w^3 \notin L_A$.

Let $f : \omega \to \{1, 2\}$ be a function such that $w^k = \Pi_n w_n^{3\,[f(n)=k]}$ for $k = 1, 2$. (As on page 32, $w^{[\phi]}$ denotes w if ϕ, ϵ otherwise.)

Let $q_0^3 = q_I$, $q_{n+1}^3 = \delta(q_n^3, w_{n+1}^3)$. For $k = 1, 2$ let $q_0^k = q_I$, $q_{n+1}^k = \delta(q_n^k, w_{n+1}^k)$ if $f(n+1) = k$, and q_n^k otherwise. Let $S^k = \limsup q^k$ for $k = 1, 2, 3$.

Since $w^1, w^2 \in L_A$ and $w^3 \notin L_A$, we have that S^1 and S^2 are both even, but S^3 is not. It can be easily shown that there exist some a, b such that for all $k = 1, 2, 3$ we have $q^k(a) = q^k(b)$, and $\exists m \in \{a \ldots b\} \operatorname{rank}(q_m^k) = S^k$.

Let $q_k = q^k(a)$ and $n_k = S^k$ for $k = \{1, 2, 3\}$. It can be easily seen that our hypothesis holds. ∎

Proof of Proposition 6.12 As we can see, to determine if L_A is convex it is enough to compute the predicate P and check the condition given in Lemma 6.14. Now, P satisfies the following rules: ($\vee$ means maximum, where $\perp$ is smaller than everything else)

- (1) For each q_1, q_2, q_3, $P(q_1, q_1, \perp,\ q_2, q_2, \perp,\ q_3, q_3, \perp)$;
- (2) For each $q_1, r_1, n_1,\ q_2, r_2, n_2,\ q_3, r_3, n_3, c$, if the predicate P satisfies $P(q_1, r_1, n_1,\ q_2, r_2, n_2,\ q_3, r_3, n_3)$ and $\delta(r_1, c) = s_1$ and $\delta(r_3, c) = s_3$ then $P(q_1, s_1, n_1 \vee \operatorname{rank}(s_1),\ q_2, r_2, n_2,\ q_3, s_3, n_3 \vee \operatorname{rank}(s_3))$.
- (3) For each $q_1, r_1, n_1,\ q_2, r_2, n_2,\ q_3, r_3, n_3, c$; if the predicate P satisfies $P(q_1, r_1, n_1,\ q_2, r_2, n_2, q_3,\ r_3, n_3)$ and $\delta(r_2, c) = s_2$ and $\delta(r_3, c) = s_3$ then $P(q_1, r_1, n_1,\ q_2, s_2, n_2 \vee \operatorname{rank}(s_2),\ q_3, s_3, n_3 \vee \operatorname{rank}(s_3))$.

Rule (1) corresponds to taking ϵ as the word w_3 from Definition 6.13, and rules (2) and (3) correspond to adding one letter c to w_1 and w_2, respectively.

Now, the algorithm of computing P is as follows: whenever we discover that $P(q_1, r_1, n_1,\ q_2, r_2, n_2,\ q_3, r_3, n_3)$ for some parameters, we close it under (2) and (3); our initial knowledge is given by (1). If $P(q_1, r_1, n_1,\ q_2, r_2, n_2,\ q_3, r_3, n_3)$, then our algorithm will find it out — by using a sequence of applications of rules (1), (2) and (3) which corresponds to the words w_1, w_2, w_3 (from Definition 6.13). Also, if our algorithm finds out that $P(q_1, r_1, n_1,\ q_2, r_2, n_2,\ q_3, r_3, n_3)$, we can reconstruct the words w_1, w_2, w_3 by analyzing the sequence of applications of rules which our algorithm used. ■

Chapter 7

Unions of Half-positional Winning Conditions

In Theorem 3.7 we have shown that a union of any half-positional winning condition and a Büchi winning condition is half-positional. In Proposition 4.4 we have shown that a union of concave winning conditions is also concave and thus also half-positional. In Proposition 6.8 we have shown that a union of finitely many monotonic conditions is also monotonic and thus also half-positional. It is a known fact that Rabin winning conditions, which are finite unions of parity conditions, are half-positional [Kla92].

All these facts suggest that the following holds.

Conjecture 7.1 *Let $\mathcal{W}$ be a (finite, countable, ...) family of (finitely) half-positional winning conditions. Then $\bigcup \mathcal{W}$ is a (finitely) half-positional winning condition.*

This conjecture, which was one of the main motivations of our research, is still an open problem. Note again that we assume prefix independence here. It is very easy to find two prefix dependent winning conditions which are positionally determined, but their union is not half-positionally determined.

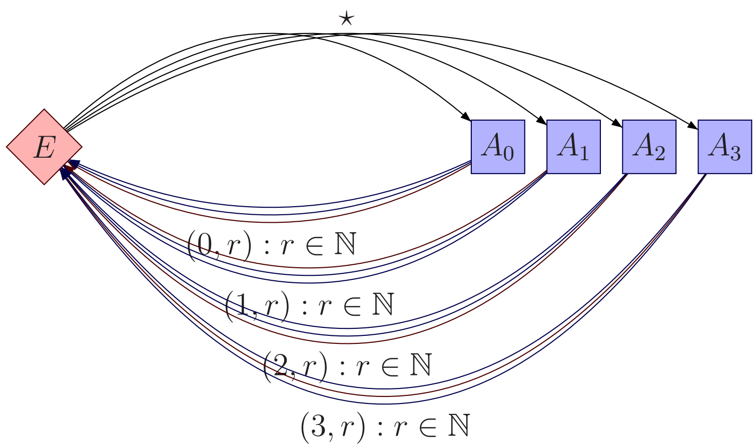
In the first section, we show that this conjecture fails for non-countable unions and infinite arenas, even for such simple conditions as Büchi and co-Büchi conditions. In the second section, we present a broad class of winning conditions which is closed under countable union, and includes some of the previously mentioned winning conditions. In the third section we present a yet broader class of winning conditions, which has even more closure properties (although is known to be closed only under finite union). In the last section we show one more example where this conjecture holds — a union of a monotonic and a concave condition.

7.1 Uncountable Unions

Theorem 7.2 *There exists a family of 2^ω Büchi conditions such that its union is not a half-positionally determined winning condition.*

There exists a family of 2^ω co-Büchi conditions such that its union is not a half-positionally determined winning condition.

Proof



Let $I = \omega^\omega$.

Our arena G over $C = \omega \times \omega \cup \{\star\}$ consists of one Eve's position E and infinitely many Adam's positions $(A_n)_{n \in \omega}$. In E Eve can choose $n \in \omega$ and go to A_n by move $E \xrightarrow{\star} A_n$. In each A_n Adam can choose $r \in \omega$ and return to E by move $A_n \xrightarrow{(n,r)} E$.

For each $y \in I$, let $S_y = \{(n, y_n) : n \in \omega\} \subseteq C$, and $S'_y = C - S_y - \{\star\}$. Let $WA_1 = \bigcup_{y \in I} WB_{S_y}$, $WA_2 = \bigcup_{y \in I} WB'_{S'_y}$.

The games (G, WA_1) and (G, WA_2) are not half-positionally determined. Let (n_k) and (r_k) be n and r chosen by Eve and Adam in the k-th round, respectively. If Eve always plays $n_k = k$, she will win both the conditions WB_{S_y} and $WB'_{S'_y}$, where $y_k = r_k$. However, if Eve plays with a positional strategy $n_k = n$, Adam can win by playing $r_k = k$. ■

7.2 Positional/suspendable Conditions

Definition 7.3 *A* **suspendable winning strategy** *for player X is a pair (s, Σ), where $s : \mathrm{Play}_X \to \mathrm{Mov}$ is a strategy, and $\Sigma \subseteq \mathrm{Play}_F$, such that:*

- *s is defined for every finite play π such that* $\mathrm{target}(\pi) \in \mathrm{Pos}_X \cap \mathrm{Win}_X$*, where* Win_X *is X's winning set;*

- *every infinite play π that is consistent with s from some point*[1] *t has a prefix π' longer than t such that $\pi' \in \Sigma$ and* $\mathrm{target}(\pi') \in \mathrm{Win}_X$*;*
- *Every infinite play π that has infinitely many prefixes in Σ is winning for X.*

We say that a player X has a **suspendable winning strategy in** *$M \subseteq G$ iff he has a suspendable winning strategy and $M \subseteq \mathrm{Win}_X$.*

Intuitively, if at some moment X decides to play consistently with s, the play will eventually reach Σ; Σ is the set of moments when X can temporarily suspend using the strategy s and return to it later without a risk of ruining his or her victory, as long as the play did not leave X's winning set.

A suspendable winning strategy is a winning strategy from Win_X, because from the conditions above we know that each play which is always consistent with s has infinitely many prefixes in Σ, and thus is winning for X.

Definition 7.4 *A winning condition W is* **positional/suspendable** *if for each arena G in the game (G, W) Eve has a positional winning strategy from her winning set Win_E and Adam has a suspendable winning strategy in his winning set Win_A, and $\mathrm{Win}_E \cup \mathrm{Win}_A = \mathrm{Pos}$.*

Example 7.5 *The co-Büchi condition WB'_S is positional/suspendable.*

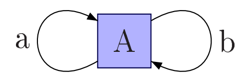
Proof Adam wants to reach colors from the set S infinitely often. We know that both players have positional winning strategies in their winning sets. Adam's suspendable winning strategy in Win_A is (s, Σ), where s is his positional winning strategy, and $\pi \in \Sigma$ iff $\mathrm{rank}(play_{|\pi|}) \in S$.

We know that if Adam plays consistently with s from some moment, then he eventually reaches S, which means that he can suspend using s and do whatever he wants. If the play does not leave Win_A, he can decide to continue using s and reach S again. If he repeats suspending and continuing infinitely many times, S is reached infinitely many times, thus Adam wins. ■

Example 7.6 *The Büchi condition WB_S for $\emptyset \subsetneq S \subsetneq C$ is not positional/suspendable.*

[1] That is, for each prefix u of π which is longer than t and such that $\mathrm{target}(u) \in \mathrm{Pos}_X$, the next move is given by $s(u)$.

Proof



Without loss of generality, $C = \{\mathbf{a}, \mathbf{b}\}, S = \{\mathbf{a}\}$. Adam has a winning strategy from each position in the arena above. However, he has no suspendable strategy: if he suspends s infinitely many times, it is possible that the play used the move of color $\mathbf{a}$ infinitely times while s „was not watching", which means that Eve wins. ■

Although no Büchi conditions, and thus no parity conditions WP_n for $n > 1$, are positional/suspendable, winning conditions with this property are common. Some of conditions which we have previously shown to be half-positional are actually positional/suspendable.

Theorem 7.7 *Let $C = [0, 1]$, $A = [0, 1/2)$. The condition $WF(A) = \{w : \limsup P_n(w) < 1/2\}$ given in Theorem 5.9 is positional/suspendable.*

Proof Consider Adam's strategy given in the proof of 5.9. That strategy led us to a word $w = w_1 w_2 w_3 \ldots$, where $P(w_i) > a_i$. If we allow an initial segment to be played not according to this strategy, we will get a word $w = u w_i w_{i+1} \ldots$ instead. Still, there will be a t such that $P_t(w) > a_i$; and we can suspend at time t. Thus, $\limsup P_n(w)$ is still at least $1/2$. ■

Note that $WF(A_1) \cup WF(A_2)$ usually is not equal to $WF(A_1 \cup A_2)$, so a union of positional/suspendable conditions given above usually is not of form $WF(A)$ itself.

Theorem 7.8 *Any monotonic condition (Theorem 6.6) is positional/suspendable.*

Proof Again, Adam's strategy given in the proof of 6.6 is suspendable, because he can suspend his strategy after each step of the iteration. Co-Büchi condition is a special case of this. ■

For the next theorem, we need the following lemma.

Lemma 7.9 *Let W be a winning condition. Suppose that, for each non-empty arena G, either there exists a non-empty subset $M \subseteq G$ where Eve has a positional winning strategy from M, or Adam has an suspendable winning strategy everywhere. Then W is positional/suspendable.*

Proof Let G be an arena. From Lemma 3.5 for half-positional winning conditions we know that W is half-positional, and $\mathrm{Pos} = \mathrm{Win}_E \cup \mathrm{Win}_A$. Let G' be the subarena with positions $\mathrm{Pos}'_X = \mathrm{Pos}_X \cap \mathrm{Win}_A$ and all moves between this positions. From our hypothesis we know that Adam has a suspendable winning strategy everywhere in G'. This strategy is also a suspendable winning strategy in Win_A in G. ■

This lemma could also be proven in a different way. Our definitions allow us to define new determinacy types D (see page 15), of (finitely) positional/suspendable winning conditions. By methods similar to Theorem 3.3 we can show that such D's are natural, and thus, Lemma 3.5 holds for them. Lemma 7.9 is then a special case of Lemma 3.5. However, theorem 3.7 does not, as $\emptyset$ is a positional/suspendable winning condition, while the Büchi condition is not.

Theorem 7.10 *A union of countably many positional/suspendable conditions is also positional/suspendable.*

Proof of Theorem 7.10 Let $\{W_1, W_2, \ldots\}$ be a countable set of positional/suspendable conditions. We will use Lemma 7.9.

If for some i we have $M \subseteq \mathrm{Win}_E(G, W_i)$, then Eve also wins from M in $(G, \bigcup_i W_i)$ as well, by using the same positional strategy.

Now assume that, for each i, we have $\mathrm{Win}_E(G, W_i) = \emptyset$, hence for every i, Adam has a suspendable strategy (s_i, Σ_i) in (G, W_i). We will define a suspendable Adam's strategy (s, Σ) winning everywhere in $(G, \bigcup_i W_i)$.

Let $(i_k)_{k\in\omega}$ be a sequence where every index i appears infinitely often. By induction on the length of play π, we define $s(\pi)$, as well as whether $\pi \in \Sigma$ or not. Let π be a play whose exactly k proper prefixes are in Σ. Then, $s(\pi) = s_{i_k}(\pi)$, and $\pi \in \Sigma$ iff $\pi \in \Sigma_{i_k}$.

Intuitively, the strategy of Adam is to first play consistently with s_{i_1} until Σ_{i_1} happens, then (after a possible suspension) play consistently with s_{i_2} until Σ_{i_2} happens, and so on. Since every Σ_i happens infinitely many times (because every index appear infinitely often in $(i_k)_{k\in\omega}$), Adam wins each W_i, and thus wins $\bigcup_i W_i$. ■

7.3 Extended Positional/suspendable Conditions

In this section we present a class of half-positional winning conditions which generalizes both positional/suspendable conditions and Rabin conditions.

Definition 7.11 *The class of* **extended positional/suspendable** *(XPS for short) conditions over C is the smallest set of winning conditions that contains all Büchi and positional/suspendable conditions, is closed under intersection with co-Büchi conditions, and is closed under finite union.*

This class contains most of half-positional winning conditions mentioned in this thesis. Using the given operations, we can obtain Büchi and co-Büchi conditions, parity conditions (inductively by taking a union with Büchi, or intersection with co-Büchi condition), Rabin conditions (by taking a finite union of parity conditions), monotonic conditions, and so on. Actually, all the specific winning conditions which have been proven in this thesis to be infinitely half-positional are XPS conditions.

Theorem 7.12 *All XPS conditions are half-positional.*

The proof is a modification and generalization of proof of half-positional determinacy of Rabin conditions from [Gra04].

Proof Let W be an XPS condition. The proof is by induction over construction of W.

We know that Büchi conditions and positional/suspendable conditions are half-positional.

If W is a finite union of simpler XPS conditions, and one of them is a Büchi condition WB_S, then $W = W' \cup WB_S$. Then W' is half-positional since it is a simpler XPS condition, and from Theorem 3.7 we get that W is also half-positional.

Otherwise, $W = W' \cup \bigcup_{k=1}^{n}(W_k \cap WB'_{S_k})$, where W' is a positional/suspendable condition, W_k is a simpler XPS condition, and WB'_{S_k} is a co-Büchi condition. (It is also possible that there is no W', but it is enough to consider this case since it is more general. A union of a finite number of positional/suspendable conditions is also positional/suspendable by Theorem 7.10.) To apply Lemma 3.5 we need to show that either Eve has a positional winning strategy from some position in the arena, or Adam has a winning strategy everywhere.

If Eve has a winning strategy from some position in (G, W'), then she has a positional strategy, and the same strategy is winning in (G, W), which is what we need.

For $m = 1, \dots, n$ let $W^{(m)} = W' \cup W_m \cup \bigcup_{k \neq m}(W_k \cap WB'_{S_k})$. We know that $W^{(m)}$ is half-positional since it is a simpler XPS condition.

Let G be an arena.

Let P_m be the set of S_m-moves, i.e., moves in G with colors from S_m. $A_m = \mathrm{source}(P_m) \cap \mathrm{Pos}_A$ is the set of Adam's positions from which he can immediately make a S_m-move. $B_m = \mathrm{Attr}_A(A_m)$ is the set of Adam's positions from which he has a strategy to reach A_m. Now, let H_m be the subgraph of G obtained by removing all the positions in B_m, and all the moves in P_m.

If Eve has a winning strategy from some position v in $(H_m, W^{(m)})$, then she also has a positional strategy, and she can use the same strategy in (G, W) — since the play is in H_m, no S_m-moves will be made during the infinite play, thus she will also win W. (Adam is unable to exit H_m, since all the positions from which he would be able to do so have been removed.)

Assume that Eve has no winning strategy from any position in the game $(H_m, W^{(m)})$, and no winning strategy from any position in (G, W'). Then Adam has the following winning strategy in (G, W).

- Since $\mathrm{Win}_E(G, W') = \emptyset$, we have $\mathrm{Win}_A(G, W') = \mathrm{Pos}$, and since W' is positional/suspendable, Adam has a suspendable winning strategy (s, Σ) in the game (G, W'). Adam uses s until the play reaches Σ.

- For $m = 1, \ldots, n$:

  - Let v be the current position.

  - $(\star)$ If $v \in H_m$ then Adam uses his winning strategy s'_m in the game $(H_m, W^{(m)})$. (Adam forgets what has happened so far in order to use s'_m.) If Eve never makes a move which does not belong to H_m then Adam wins. Otherwise, he stops using s'_m after a move p out of H_m is made.

  - $(\star\star)$ There are two cases: $\mathrm{rank}(p) \in S_m$ or $\mathrm{target}(p) \in B_m$. In the second case, in B_m, we know that Adam has a strategy which forces reaching A_m; Adam uses this strategy. Then, in A_m, Adam uses a S_m-move. (Thus, in both cases, a S_m-move is made.)

- Repeat.

If ultimately the game remains in the step $(\star)$ of the strategy above for some m, then Adam wins since he is using a winning strategy in $(H_m, W^{(m)})$. Otherwise, Adam wins W' (since he correctly resumed using his suspendable strategy in (G, W') infinitely many times) and all the co-Büchi conditions WB'_{S_m} for $m = 1, \ldots, n$ (since a S_m-move is always done in the step $(\star\star)$), hence he also wins $W \subseteq W' \cup \bigcup_{k=1}^{n} WB'_{S_k}$. ∎

7.4 Combining Concave and Monotonic Conditions

In this section we investigate how our conjecture about unions of half-positional conditions works for concave (Chapter 4) and monotonic (Section 6.2) conditions.

In the beginning of this chapter we have noted that arbitrary unions of concave conditions and finite unions of monotonic conditions are also in these classes, and thus are also half-positional. A countable union of monotonic conditions is not necessarily defined by a single monotonic automaton, but, from Theorem 7.10, it is still positional/suspendable; however, a union of cardinality 2^ω of monotonic conditions need not be half-positionally determined, as shown by Theorem 7.2 (co-Büchi conditions are monotonic).

As a conclusion of this chapter, we will show the following theorem, which solves the union problem for a union of a monotonic and a concave winning condition. Since both concave and monotonic winning conditions are closed under finite union, we obtain that Conjecture 7.1 is true for the class of winning conditions containing all monotonic and concave winning conditions, and closed under finite union.

Theorem 7.13 *Let $W_1 \subseteq C^\omega$ be a concave winning condition, and A be a monotonic automaton. Then the union $W = W_1 \cup WM_A$ is a half-positionally finitely determined winning condition.*

Proof Let $G = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$. A proof by induction on $|\mathrm{Mov}|$. We define v, $M = M_1 \cup M_2$, G_1, G_2, Win_i^A, and t_i exactly like in the proof of Theorem 4.7. We will show a winning strategy for Adam in the set Win_1^A in the case when $v \in \mathrm{Win}_1^A = \mathrm{Win}_2^A$; all other cases are done just like in the proof of Theorem 4.7.

Let n be the accepting state of A, and $q_k(n, c) = n$ for each $c \in C$. For a play π, we define sequences of states $(q_k(\pi))_k$ and $(r_k(\pi))_k$ by induction: $q_0(\pi) = 0$; $r_k(\pi) = 0$ if $q_k(\pi) = n$ and $\mathrm{source}(\pi_{k+1}) = v$, and $q_k(\pi)$ otherwise; and $q_{k+1}(\pi) = \delta(r_k(\pi), \mathrm{rank}(\pi_{k+1}))$. We can see that if the play π visits v infinitely many times, then $\mathrm{rank}(\pi) \notin WM_A$ iff $q_k(\pi) = n$ for infinitely many values of k.

Let $K = \mathrm{dom}\pi$. Let $S_v = \{k \in K : \mathrm{source}(\pi_k) = v\}$. We define the function $f : K \to \{1, 2\} \times Q$ in the following way. If $k < \min S_v$, we take $f(k) = (1, 0)$. Otherwise, let k' be the greatest element of S_v such that $k' \le k$, and $f(k) = (i, q_{k'}(\pi))$ where $\pi_{k'} \in M_i$.

Let $\pi_{(i,q)} = \Pi_{k \in K} \pi_k^{[f(k)=(i,q)]}$. Thus, we have cut the play π into segments which start and end in v (except possibly the last infinite one), and presented

π as a shuffle of plays $\pi_{(i,q)}$. (A shuffle of more than two words is defined in an obvious way.)

Now, we can define Adam's strategy: for a finite play π of length m, let $t(\pi) = t_j(\pi_{(j,q)})$, where $(j,q) = f(m)$. If Adam consistently plays with the strategy t then, for each $i = 1, 2$ and each $q \in Q$, all plays $\pi_{(i,q)}$ are consistent with t_i.

We check that t is indeed a winning strategy for Adam in the set Win_1^A. Let π be an infinite play consistent with t; we have to show that $\mathrm{rank}(\pi) \notin W_1$ and $\mathrm{rank}(\pi) \notin WM_A$.

For each (i,q) we have $\mathrm{rank}(\pi_{(i,q)}) \notin W_1$, since this play is consistent with t_i, which is a winning strategy. Hence, from concavity of W_1, and the fact that π is a shuffle of $\pi_{(i,q)}$ for all $i = 1, 2$ and $q \in Q$, we get that $\mathrm{rank}(\pi) \notin W_1$.

Let $S \in \{1, 2\} \times Q$ be the set of all (j,q) such that $f(m) = (j,q)$ for infinitely many values of m. Let (j_s, q_s) be the element of S with the greatest value of q_s. Assume $q_s < n$, otherwise $\mathrm{rank}(\pi) \notin WM_A$ is obvious.

Adam wins the play $\pi' = \pi_{(j_s,q_s)}$ since it is consistent with t_{j_s}. The play π' is infinite. Let $S'_v = \{k \in \omega : \mathrm{source}(\pi'_k) = v\}$. If S'_v is finite, this means that π and π' have a common suffix (as we don't return to v we are stuck in $\pi_{(j_s,q_s)}$), and from the prefix independence of WM_A Adam wins π. Otherwise π' visits v infinitely many times, and hence $q_k(\pi') = n$ for infinitely many values of k.

For $m \in S'_v$ let $m^+ = \min\{m' \in S'_v : m' > m\}$, and P_m be the segment of play from $m+1$-th to m^+-th move.

Let M be the set of m's such that $q_{m^+}(\pi') = \delta(r_m(\pi'), P_m) > q_s$ and $r_m(\pi') \leq q_s$. Since $\mathrm{rank}(\pi') \notin WM_A$, and thus $q_m(\pi')$ reaches n infinitely many times, after which $r_m(\pi')$ is reset to 0, the set M is infinite.

Each segment P_m appears also in play π after some m'-th move, where $q_{m'}(\pi) = q_s$. For $m \in M$, after $m' + |P_m$ moves of play π, we are back in v, and we have

$$q_{m'+|P_m|}(\pi) = \delta(q_{m'}(\pi), \mathrm{rank}\, P_m) = \delta(q_s, \mathrm{rank}\, P_m) \geq \delta(r_m(\pi'), \mathrm{rank}\, P_m) > q_s.$$

Hence, we have found that, in π, after $m' + |P_m|$ moves, we are back in v, with the automaton state greater than q_s. Since this is true for each m in the infinite set M, we are back in v with the automaton state greater than q_s infinitely many times, which contradicts the definition of S. ■

Chapter 8

Beyond Positional Strategies

When it is impossible to win the game with a positional strategy, it is still possible that we can win using a strategy which is not positional, but has some other, weaker property. In this chapter we present two kinds of such strategies. We answer some questions regarding these strategies, but it is currently an area of research and many questions remain open.

The first kind is strategies with memory. When it is impossible to win the game using no memory, we can still hope to use the smallest possible amount of memory states. We present two kinds of memory: normal ("chaotic") and chromatic. We estimate memory required by the other player for the winning conditions which were introduced before.

The second kind is persistent strategies, which are "almost" positional.

8.1 Strategies with Memory

Definition 8.1 *A* **memory** *for a game* (G, W) *is a pair* $\mathcal{M} = (M, \mu)$*, where* M *represents possible memory states, and* $\mu : M \times \mathrm{Mov} \to M$ *is the* **memory update function**. *We extend* μ *as usual to* $\mu : M \times \mathrm{Mov}^* \to M$*.*

A **strategy with memory** $\mathcal{M}$ *is a function* $\widehat{s} : \mathrm{Pos}_X \times M \to \mathrm{Mov}$*. We say that* $\widehat{s}$ *is* **winning** *from position* v *and* **initial memory state** m *iff the strategy* $\widehat{s}_m$ *given by* $\widehat{s}_m(\pi) = \widehat{s}(\mathrm{target}(\pi), \mu(m, \pi))$ *is winning from* v*. We say that* $\widehat{s}$ *is* **winning** *from position* v *iff it is winning from* **each** *initial memory state* $m \in M$*.*

The usual definition of memory and a strategy with memory from literature, e.g., in [DJW97], where memory required to win a game with a Müller winning condition is calculated, is a bit different: initial memory state is declared in the memory. We have decided to force our strategy with memory to win from all memory states. According to the following proposition, the choice of definition does not matter as far as the winning sets are concerned, because if our strategy wins from some position from only some memory states, then it can be fixed to win from all of them.

Proposition 8.2 *Let (G, W) be a game. Suppose that $\widehat{s}$ is strategy for X with memory $\mathcal{M}$ winning from a position $v \in \mathrm{Pos}$ and memory state m. Then there exists a strategy $\widehat{s}'$ for X with memory $\mathcal{M}' = (M, \mu')$ winning from v and each memory state.*

Proof Let $S \subseteq M$ be the set of memory states m' for which there is a play π consistent with $\widehat{s}_m$, which goes through v in memory state m' (i.e., there is a prefix π' of π such that $\mathrm{target}(\pi') = v$ and $\mu(m, \pi') = m'$). From prefix independence we get that $\widehat{s}$ is also winning from v from each memory state $m' \in S$.

For $m' \notin M$, we change our strategy in v in the following way: $\widehat{s}'(v, m') = \widehat{s}(v, m)$; $\mu'(v, m') = \mu(v, m)$. For other positions and memory states $\widehat{s}'$ and μ' behave exactly like $\widehat{s}$ and μ.

It can be easily seen that each play consistent with $\widehat{s}'_{m'}$ for $m' \in S$ will be also consistent with $s_{m'}$, and each play consistent with $\widehat{s}'_{m'}$ for $m' \notin S$ will be consistent with s_m. Since these strategies are winning, $\widehat{s}'$ is also a winning strategy from v. ∎

Our definitions allow us to construct new determinacy types, which require one or both players to have a strategy with memory of finite size, or a strategy with memory of size n. These determinacy types are natural (thanks to our choice of definition), and additionally, the claim of Theorem 3.7 is also true for them.

Definition 8.3 *For a winning condition W and player X, let $\mathrm{mm}_X(W)$ be the smallest n such that for each arena G the player X can win in $\mathrm{Win}_X(G)$ using a strategy with memory of size n.*

Example 8.4 *Let WQ be the winning condition from the proof of Proposition 4.9, and G be an arena with one Adam's position and two moves, which are colored 0 and 1. Then Adam has a winning strategy in (G, WQ), but has no winning strategy with memory of any finite size.*

Positive examples will be shown in the next section.

8.2 Chromatic Memory

As we can see, this standard definition of memory is strongly dependent on the arena. Since in this thesis we are interested in properties of winning conditions rather than games, we need memory which could be defined regardless of the arena. As we will see, such *chromatic* memory has some nice properties. Below is the natural definition.

Definition 8.5 *We say that a memory $\mathcal{M}$ is* **chromatic** *if it depends only on colors of the moves, i.e. there exists a function $\widehat{\mu} : M \times C \to M$ such that $\mu(m, p) = m$ when $\mathrm{rank}(p) = \epsilon$, and $\mu(m, p) = \widehat{\mu}(m, \mathrm{rank}(p))$ otherwise.*

A **strategy with chromatic memory** *$(M, \widehat{\mu})$ is a strategy with memory (M, μ) where μ is a chromatic memory given by $\widehat{\mu}$. As usual, we extend $\widehat{\mu}$ to $\widehat{\mu} : M \times C^* \to M$.*

Proposition 8.6 *Let W be an ω-regular winning condition, recognized by a strongly connected DFA $A = (Q, q_I, \delta, \mathrm{rank})$.*

Then both players have strategies with chromatic memory $\mathcal{M} = (Q, \delta)$ in their winning sets.

Proof Positional winning strategies in the game (G^1, WP_d) defined in the proof of Theorem 6.9 can be interpreted as strategies with such a chromatic memory. Since A is strongly connected, the initial memory state is indeed irrelevant (Proposition 6.2). ■

Proposition 8.7 *Let $W = WM_A$ be a monotonic winning condition, where $A = (n, \delta)$. Then Adam has a winning strategy with chromatic memory $\mathcal{M} = (\{0, \ldots, n-1\}, \widehat{\mu})$ in his winning set, where $\widehat{\mu}(k, c) = \delta(k, c) \bmod n$.*

Proof Adam's strategy given in the proof of Theorem 6.6 can be interpreted as a strategy with such a chromatic memory. ■

Definition 8.8 *For a winning condition W and player X, let $\mathrm{mm}^{\chi}_X(W)$ be the smallest n such that for each arena G X can win in $\mathrm{Win}_X(G)$ using a strategy with* **chromatic** *memory of size n.*

Determinacy types which require using a certain chromatic memory structure $\mathcal{M}$ are natural, and hypothesis of Theorem 3.7 also works for them. (If we require a chromatic memory of given size, we don't get a natural determinacy type, as using different chromatic memory structures in various parts of the arena breaks the globalization condition for them.)

The following simple proposition gives a nice property of mm^{χ}_X.

Proposition 8.9 *Let W be a winning condition such that $\mathrm{mm}^{\chi}_X(W) = n$. Then there is a single chromatic memory $\mathcal{M}_W$ of size n such that, for each arena G, and each starting position v_0 in G, X can win in $\mathrm{Win}_X(G)$ using $\mathcal{M}_W$.*

Proof Let T be the set of all chromatic memories $\mathcal{M}$, where $\mathcal{M} = (M, \mu)$ and $M = \{0, \ldots, n-1\}$. This set is finite and it contains all possible memories of size n (up to isomorphism).

For an arena G and a position v in $\mathrm{Win}_X(G)$, let $U(G, v) \subseteq T$ be the set of all memories $\mathcal{M} \in T$ such that X can win in G from v using $\mathcal{M}$. Since $\mathrm{mm}^{\chi}_X(W) = n$, and T contains all possible chromatic memories, we know that $U(G, v)$ is non-empty.

We have to show that there exists a chromatic memory $\mathcal{M}_W$ such that for all G and v we have $\mathcal{M}_W \in U(G, v)$.

Assume to the contrary that for each $\mathcal{M} \in T$ there exists an arena $G(\mathcal{M})$ and a winning position $v(\mathcal{M})$ such that $\mathcal{M} \notin U(G(\mathcal{M}), v)$. We can assume that $G(\mathcal{M}_1)$ and $G(\mathcal{M}_2)$ are disjoint for $\mathcal{M}_1 \neq \mathcal{M}_2$.

Let $G = \bigcup_{\mathcal{M} \in T} G(\mathcal{M})$, and G_0 be G plus one additional position v_0 from which X's opponent can choose to go to any position in $\mathrm{Win}_X(G)$. Still, we have that $U(G_0, v_0)$ is non-empty; let $\mathcal{M} \in U(G_0, v_0)$.

We have a contradiction — in x_0 X's opponent can decide to go to $v(\mathcal{M})$ in $G(\mathcal{M})$, and we know that X has no winning strategy using $\mathcal{M}$ from $v(\mathcal{M})$. ■

If we restrict ourselves to A-arenas, we have the following results. A-arenas (see Section 2.5) are arenas such that colors appear in (all) positions instead of edges. Equivalently, for each position v, each move from v has the same color $\mathrm{rank}(v) \neq \epsilon$.

Proposition 8.10 *Let W be a winning condition over C, and $\mathcal{M} = (M, \widehat{\mu})$ be a chromatic memory.*

Let $W \times \mathcal{M}$ be a winning condition over $C \times M$ such that

$$(c_0, m_0)(c_1, m_1)(c_2, m_2) \ldots \in W \times \mathcal{M}$$

iff the following two conditions are satisfied: $c_0 c_1 c_2 \ldots \in W$, and, for almost all k's, we have $\widehat{\mu}(m_k, c_k) = m_{k+1}$.

Then Eve can win in her winning set in each A-arena using chromatic memory $\mathcal{M}$ iff $W \times \mathcal{M}$ is A-half-positional.

Proof First, suppose that $W \times \mathcal{M}$ is A-positional. Let G be an A-arena. Create the new arena G' such that $\mathrm{Pos}'_X = \mathrm{Pos}_X \times M$, and for each $m \in M$ and move $v \xrightarrow{c} w \in \mathrm{Mov}$, we have a move $(v, m) \xrightarrow{(m,c)} (w, \widehat{\mu}(m, c))$ in Mov'.

It can be easily seen that, for any $m \in M$, Eve wins a play π in (G, W) from position v iff she wins its corresponding play in $(G', W \times \mathcal{M})$ from position (v, m). Thus, we have $\mathrm{Win}_X(G', W \times \mathcal{M}) = \mathrm{Win}_X(G, W) \times M$. Also, if Eve has a positional strategy in $(G', W \times \mathcal{M})$ from (v, m) for all m, it can be interpreted as a strategy with chromatic memory $\mathcal{M}$ from v in G.

Now, suppose that Eve can win in her winning set in each A-arena using $\mathcal{M}$. We will show that $W \times \mathcal{M}$ is A-half-positional, using Lemma 3.5 for A-half-positional winning conditions: we will show that, for each arena G, either Adam has a winning strategy everywhere in $(G, W \times \mathcal{M})$, or Eve has a winning positional strategy from some position v_0.

Let G be an arena. Let v_0 be a position from which Eve has a winning strategy s in game $(G, W \times \mathcal{M})$ such that for each play π of rank $(c_0, m_0)(c_1, m_1)(c_2, m_2)\ldots$ starting in v_0 and consistent with s we have, for each k, $\widehat{\mu}(m_k, c_k) = m_{k+1}$.

If there is no such v_0, then Adam wins everywhere. Indeed, it means that in each position Adam has a strategy t to either fail W, or to force (*) $\widehat{\mu}(m_k, c_k) \neq m_{k+1}$ for some k. Thus, if Adam restarts his strategy t after each (*), he will win.

Let G_1 be a subarena of all positions and moves which can be used in a play starting in v_0 consistent with s. Let G_2 be the arena over C obtained from G_1 by replacing each color (c, m) by c; it can be easily seen that a play π in $(G_1, W \times \mathcal{M})$ is winning iff its respective play in (G_2, W) is winning ($\widehat{\mu}(m_k, c_k) = m_{k+1}$ is guaranteed by consistency with s). Since Eve has a winning strategy in G_1 from v_0, she also has a winning strategy in G_2 from v_0, and by assumption, in G_2 she has a winning strategy from v_0 with chromatic memory M, $\widehat{s}$.

Now, in G_1 she can use the positional strategy $s(v) = \widehat{s}(v, m)$, where $\mathrm{rank}(v) = (c, m)$. (Note that we can define a strategy in such a way only for A-arenas.) We can see that this strategy is winning. ■

Proposition 8.11 *Define* $\mathrm{mm}_E^{\chi,A}(W)$ *exactly like* $\mathrm{mm}_E^{\chi}(W)$*, except that we restrict to A-arenas only. Then, if Conjecture 7.1 holds for A-arenas, then* $\mathrm{mm}_E^{\chi,A}(W_1 \cup W_2) \leq \mathrm{mm}_E^{\chi,A}(W_1)\mathrm{mm}_E^{\chi,A}(W_2)$*.*

Proof Let $\mathcal{M}_i$, for $i = 1, 2$, be the memory of size $\mathrm{mm}_E^{\chi}(W_i)$ such that for each A-arena G, Eve has a winning strategy in $\mathrm{Win}_E(G)$ using $\mathcal{M}_i$. (We know that such $\mathcal{M}_i$ exists from Proposition 8.9 (relativized to A-arenas).)

Define $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ in the natural way, i.e. if $\mathcal{M}_i = (M_i, \widehat{\mu}_i)$ then then $\mathcal{M} = (M, \widehat{\mu})$, where $M = M_1 \times M_2$, and

$$\widehat{\mu}((m_1, m_2), c) = (\widehat{\mu}_1(m_1, c), \widehat{\mu}_2(m_2, c)).$$

It can be easily seen that $\mathcal{M}$ is a good chromatic memory for both W_1 and W_2 (i.e. in each A-arena G Eve has a winning strategy in $\mathrm{Win}_E(G)$ using $\mathcal{M}$). Thus, by Proposition 8.10, $W_1 \times \mathcal{M}$ and $W_2 \times \mathcal{M}$ are A-half-positional. By Conjecture 7.1, their union, which is $W \times \mathcal{M}$, is also A-half-positional. Again by Proposition 8.10, $\mathcal{M}$ is good for W. ∎

Unfortunately, we have no proofs for corresponding facts for C-arenas.

8.3 Chromatic Memory Requirements

In this section, we extend Theorems 6.9 and 6.10 to calculate chromatic memory requirements for a ω-regular winning condition.

Definition 8.12 *We say that an arena G* **adheres** *to chromatic memory $\mathcal{M} = (M, \widehat{\mu})$ iff there is a function $\phi : \mathrm{Pos} \to M$ such that for each move $v \xrightarrow{c} w$ in G we have $\phi(w) = \widehat{\mu}(\phi(v), c)$.*

Theorem 8.13 *Let W be a winning condition accepted by a deterministic finite automaton with parity acceptance condition $A = (Q, q_I, \delta, \mathrm{rank} : Q \to \{0 \dots d\})$ (see Definition 6.1), and $\mathcal{M}$ be a chromatic memory. Let a* witness arena *be an arena G such that in game (G, W) from some position v_I Eve has a winning strategy, but not a winning strategy which uses $\mathcal{M}$ as memory. Then, if there exists a witness arena, then there exists a witness arena G such that G adheres to $\mathcal{M}$, and Eve has only one position in G, and only two moves from here.*

Proof Let $G_0 = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$ be a witness arena. Let $v_I \in \mathrm{Pos}$ be a position where Eve has a winning strategy, but not a strategy using $\mathcal{M}$.

Let $m_I \in M$ be the state such that no strategy using $\mathcal{M}$ is winning when starting from position v_I and memory state m_I. We can assume that such a m_I exists. Otherwise, for each m there is a strategy $\widehat{s}_m : \mathrm{Pos}_E \times M \to \mathrm{Mov}$. Suppose that the set of memory states is ordered with $<$. We can define a global winning strategy with memory M in the following way: $\widehat{s}(v, m) = \widehat{s}_{i(m)}(v, m)$, where $i(m)$ is the smallest (according to $<$) memory state in which reaching position v in memory state m is possible in a play consistent with $\widehat{s}_{i(m)}$ starting from v and $i(m)$. (If reaching v in memory state m is impossible at all, we can leave $\widehat{s}(v, m)$ undefined.) It can be easily checked that such a $\widehat{s}$ is indeed a winning strategy from v_0, contrary to our assumption that G_0 is a witness arena. (This proof is similar to the proof of globalization condition, Theorem 3.3.)

First, we construct an arena G_1 such that G_1 adheres to $\mathcal{M}$. Let $G_1 = (\mathrm{Pos}^1_A, \mathrm{Pos}^1_E, \mathrm{Mov}^1)$, where $\mathrm{Pos}^1_X = Pos_X \times M$ and

$$\mathrm{Mov}^1 = \{((v_1, m_1), (v_2, m_2), c) : (v_1, v_2, c) \in Mov \wedge \widehat{\mu}(m_1, c) = m_2\}.$$

One can easily check that G_1 is also a witness arena. (To show that G_1 adheres to $\mathcal{M}$, take $\phi(v, p) = p$.)

If there was a positional strategy in G_1 from (v_I, m_I), then we could use it to construct a strategy in G_0 using $\mathcal{M}$ from v_I and m_I; we have assumed that such strategy does not exist.

Thus, G_1 is also a witness arena against positionality of W. Thus, we can now apply to it the same simplification which we used in Theorem 6.9, obtaining a new arena G_2 with only one Eve's position v_1 and two moves, and adhering to $\mathcal{M}$, with $\phi(v_1) = m_I$ (it can be seen that our simplification preserves adherence), and where Eve has no positional strategy from v_1.

Eve has no winning $\mathcal{M}$-strategy from position v_1 and memory state m_I. Otherwise, this strategy would be positional, since G_2 adheres to $\mathcal{M}$ — and we know that Eve has no positional winning strategy. ■

Theorem 8.14 *Let W be a winning condition accepted by a deterministic finite automaton with parity acceptance condition $A = (Q, q_I, \delta, \mathrm{rank} : Q \to \{0 \dots d\})$. Then $\mathrm{mm}^\chi_E(A)$ can be calculated in single exponential time.*

Proof We check all possible chromatic memories of size up to $|Q|$ until we find one which works. There are $O(|Q|^{|Q||C|})$ such memories, and checking whether memory works can be done in a way analogous to Theorem 6.10. ■

8.4 Chromatic Versus Chaotic

As we have seen, chromatic memories have some nice properties, which need not hold for "chaotic" non-chromatic memories. Do we lose something if we restrict ourselves to chromatic memories only? The natural question is whether whenever we can win with a memory of size n, we can also win with a chromatic memory of size n; in other words, does $\mathrm{mm}^\chi_X(W) = \mathrm{mm}_X(W)$?

In the following examples, equality indeed holds, although it is not completely trivial.

Proposition 8.15 *For the winning condition $WR = C^\omega - (C^* a^n)^\omega$ over $C = \{\mathbf{a}, \mathbf{b}\}$ we have $\mathrm{mm}_A(WR) = \mathrm{mm}^\chi_A(WR) = n$.*

Note that the winning condition given above is half-positional since it is a monotonic condition (the monotonic automaton recognizes the language $C^*a^nC^*$; see Example 6.4).

Proof The language $C^*a^nC^*$ is recognized by an automaton with states $\{0, \ldots, n\}$ (the state is equal to the number of as at the end of our word; the automaton is shown in Example 6.4). Thus, from Proposition 8.7, $\mathrm{mm}^{\chi}_{A}(WR) \leq n$.

We also know that $\mathrm{mm}_A(WR) \leq \mathrm{mm}^{\chi}_{A}(WR)$. It is enough to show an arena which requires n memory states in the non-chromatic case.

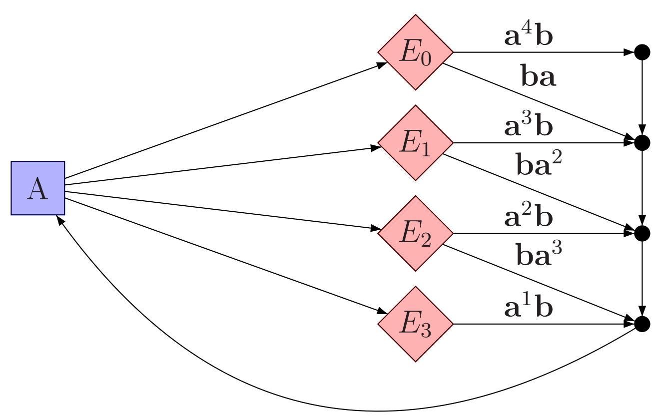
We will construct our arena from *gadgets*, i.e. subgraphs which perform required simple operations. In the sequel of this proof, by *state* of a play π we mean the current number of as at the end of $\mathrm{rank}(\pi)$.

Let s_i be a *synchronizer*, i.e. a gadget which sets the memory state to i. (This is just a sequence of moves of colors $\mathbf{ba}^i$.)

Let w_i be a *tracker*, i.e. gadget such that when w_i is entered by Adam in state $\geq i$ infinitely many times, then Adam wins. (Again, this is just a sequence of colors a^{n-i}.)

In our arena, we will have only one Adam's position, A. Adam has to decide to move to one of n Eve's positions, $E_0, \ldots, E_{n-1}$. When Adam decides to go to E_j, then Eve can decide between $w_j s_0$ and, for $j < n-1$, s_{j+1}. Both of these gadgets return to A.

The following picture shows the arena for $n = 4$.



Adam wins iff the play enters w_i in state $\geq i$ infinitely many times. Suppose that Eve uses the reasonable strategy to never choose w_i when the memory state is $\geq i$ (if she has a choice); and if both moves have this property, choose the one which sets the state to the lower value. Thus, if Eve is in E_j

in memory state i, then she will choose s_{j+1} if $j \leq i$ and $j < n-1$, and $w_j s_0$ otherwise.

We can easily see that Adam has to make use of all his possible moves from A to win against Eve's strategy given above. Thus, he needs memory of size n, since there are so many available moves. ∎

Proposition 8.16 *Let* $C = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. *Let* $W_{\mathbf{ab}} = W_{\mathbf{a}} \cap W_{\mathbf{b}}$, *where*

$$W_{\mathbf{a}} = C^\omega - (C^*(\mathbf{ab}^*)^A)^\omega \text{ and } W_{\mathbf{b}} = C^\omega - (C^*(\mathbf{ba}^*)^B)^\omega.$$

Then we have $\mathrm{mm}_A(W_{\mathbf{ab}}) = \mathrm{mm}_A^\chi(W_{\mathbf{ab}}) = AB$.

Our winning condition means that Adam wants to infinitely many times see A **a**'s or B **b**'s between two consecutive occurrences of **c**.

Note that we can easily see that $W_{\mathbf{a}}$ and $W_{\mathbf{b}}$ require memory of size A and B, respectively (indeed, they are similar to WR from 8.15, except that some colors are renamed and some new colors are added which do nothing). Thus, our memory size agrees with dual of Proposition 8.11 (we take Adam instead of Eve, and intersection instead of union).

Proof We will use the same gadget construction as in proof of Proposition 8.15. It is enough to define play states, and construct the synchronizer and tracker gadgets, which satisfy the required properties.

Play state $xB + y$ for $0 \leq x < A$, $0 \leq y < B$ corresponds to seeing x letters **a** and y letters **b** after the last c.

The synchronizer has a form of $s_{xB+y} = c\mathbf{a}^x\mathbf{b}^y$, and the tracker w_{xB+y} is an Eve's position where she can choose between $\mathbf{a}^{A-1-x}\mathbf{b}^{B-y}$ and $\mathbf{a}^{A-x}$. ∎

8.5 Persistent Strategies

Positional strategies always use the same move in the same position. So do persistent strategies — if Eve's position is visited several times during the play, then she always uses the same move. However, they have an additional power over positional strategies. Positional strategy is written down before the game, so Adam may predict Eve's future moves and adapt his strategy. Not so for persistent strategies — where such a positional strategy is not written down, and thus Eve is able to "trick" Adam that she is using a positional strategy by using always the same move in the same position, while in fact she is choosing the move during her first visit to each position. Persistent strategies have been introduced in [MT02]; the result there is that games with positive winning conditions (i.e. such that their complement is closed under shuffles with C^ω) admit persistent strategies for Eve.

Definition 8.17 *A winning strategy s is* persistent *iff $s(\pi_1)$ equals $s(\pi_1\pi_2)$ whenever* $\mathrm{target}(\pi_1) = \mathrm{target}(\pi_1\pi_2)$.

Persistence gives us new determinacy types (Section 2.4).

Definition 8.18 *A winning condition W is* **half-persistent** *if it is in determinacy type (persistent, arbitrary, infinite) (i.e. for each arena and each starting position either Eve has a persistent winning strategy, or Adam has a winning strategy). A winning condition W is* **persistent** *if it is in determinacy type (persistent, persistent, infinite). As before, we add "finitely" or "A-", "B-", "C-" when we restrict admissible arenas (Section 2.5).*

The following example shows that half-persistence is indeed a weaker property than half-positionality, at least for A-arenas and B-arenas.

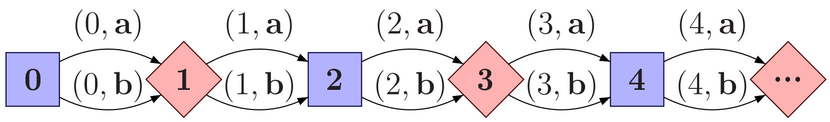
Example 8.19 *Let $C = \omega \times \{\mathbf{a}, \mathbf{b}\}$. Let $f : C \times C \to \{0, 1, 2\}$ be given by $f((n, c), (n', c')) = 2$ if $n' \neq n + 1$, $f((2n, c), (2n + 1, c')) = 1$ iff $c \neq c'$, and 0 otherwise. Let $W = \{c_1 c_2 \ldots : \limsup f(c_n, c_{n+1})$ is even$\}$. Then W is B-half-persistent and A-half-positional, but not B-half-positional nor C-half-persistent.*

Proof We will first show that W is A-half-positional. Let G be an A-arena. If we color each move $v \to w$ with $f(\mathrm{rank}(v), \mathrm{rank}(w))$, we obtain a B-arena G', and a play is winning in (G', WP_2) (WP_2 is the parity condition over $\{0,1,2\}$) iff it is winning in (G, W). We know that the (G', WP_2) is positionally determined, hence so is (G, W).

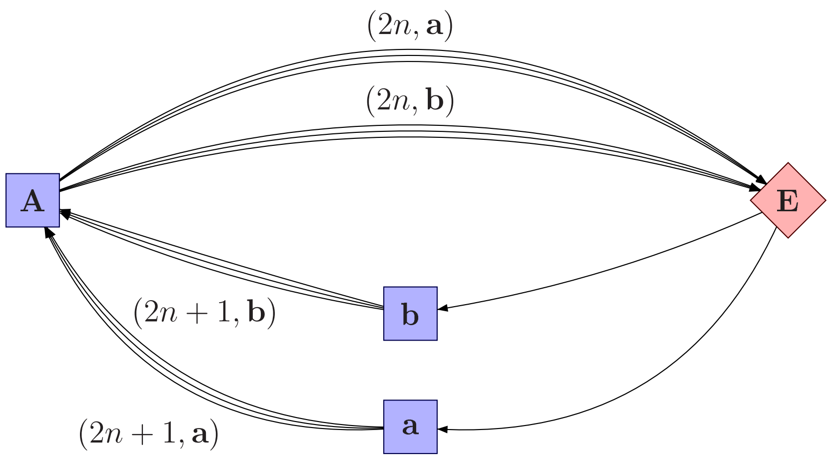
To show that W is half-persistent, we can define a strategy with (chromatic) memory which uses C as the set of memory states — it remembers the last color which appeared in our play. We can prove that a strategy s with such a memory exists using a similar reduction to WP_2.

Now, we can define our persistent strategy s': if we visit the same position again, then we do the same thing; otherwise, do what s suggests. This strategy is winning, because the only way for Adam to win is to have the first components of color growing until infinity, which means that from some moment positions do not repeat (it is important here that we are playing on a B-arena and not on a C-arena), and Eve always does what s suggests, which is a winning strategy.

W is not B-half-positional, as shown by the following infinite arena. If Eve is using a positional strategy, Adam can anticipate her choice of **a** or **b** in position $n + 1$, and choose the other letter in position n.



W is also not C-half-persistent, as shown by the following infinite arena. Each multiple arrows represents a set of moves (we take each $n \in \omega$). If Eve uses a persistent strategy, she must decide to use either letters **a** or **b**, which allows Adam to adjust his strategy to always use the other letter in position A, while the numbers grow in the correct way.



■

Note that W is half-persistent although it is not positive (in the sense of [MT02]: its complement is not closed under shuffles with C^ω).

Example 8.20 *Let C and f be like in Example 8.19. Let $W_1 = \{c_1c_2\ldots :$ $\limsup f(c_n, c_{n+3})$ is even$\}$. Then W_1 is B-half-persistent, but not even A-half-positional.*

Let W_2 be the set of words w over $C \cup \{\star\}$ such that two $\star$'s appear in a row in w infinitely many times, or the word obtained from w by removing all $\star$'s, w', satisfies $w' \in W$. Then W_2 is A-half-persistent but it is not B-half-persistent nor A-half-positional.

Proof The reasoning is similar to one used in Example 8.19. ■

We have no example of a winning condition which is C-half-persistent but not C-half-positional, nor of a winning condition which is B-half-positional but not C-half-positional. Other than that, all combinations of (A,B,C)-half-{positionality/persistence} are possible, as long as they obey obvious inclusions (i.e., $\gamma-$half-positional implies $\gamma-$half-persistent, and having a property in a bigger class of arenas implies having it in a smaller class).

Proposition 8.21 *Determinacy types introduced in this section are natural.*

Before we will be able to prove an analogue of Theorem 3.7, we need the following theorem.

Theorem 8.22 *Let D be one of determinacy types introduced in this section (which requires persistent strategies for Eve). Let W be a winning condition. Let G be a D-arena such that (G, W) is D-determined. Let s be Eve's D-strategy winning from a position u. Suppose that there is a play starting in u consistent with s where Eve uses a move p in $v \in \mathrm{Pos}_E$. Let G' be an arena which is the same as G except that all moves from v except p are removed. Then (G', W) is also D-determined, and the winning sets in G and G' are equal.*

Proof If Eve has a winning persistent strategy from w in G', then it is also a winning persistent strategy from w in G. Also, if Adam has a winning D-strategy from w in G, then it is also a winning D-strategy from w in G'. So, it is enough to show that if Eve has a winning strategy s from w in G, then she has also a winning strategy from w in G'.

Let π be a finite play starting in u after which Eve decides to use p (i.e. $\mathrm{target}(\pi) = v,\ s(\pi) = p$, and it is the first visit to v). Let M be the set of all positions which can be reached in a play of which π is a prefix and which is consistent with s.

Eve's strategy s' is as follows. Let π be a finite play. While $\mathrm{target}(\pi)$ is not in M, play the same as s: $s'(\pi) = s(\pi)$. The strategy changes if the play reaches M; suppose that happens after a play π_0. Since $\mathrm{target}(\pi_0)$ is in M, we have a finite play π_1 from u such that $\mathrm{target}(\pi_1)$ is in u, and π_1 goes through p. In the following moves, Eve acts as if the play started with π_1, not in π_0: $s'(\pi_0\pi) = s'(\pi_1\pi)$. Since no position appearing before π_0 was in M, the play will never return to a position from before π_0, so trading initial segments won't break persistence. ■

Corollary 8.23 *A winning condition is finitely half-persistent iff it is finitely half-positional.*

Proof Let G be a finite arena. We remove moves from Eve's winning set using Theorem 8.22 until only one move remains from each position. The winning sets do not change, and Eve's strategy in the result has to be positional. This positional strategy works in G. ■

Theorem 8.24 *Let D be one of the determinacy types introduced in this section or their duals. If W is D-determined, so is $W \cup WB_S$.*

Proof The proof is also analogous to Theorem 3.7, but there is one difference. In proof of Theorem 3.7, we defined a game G', where Eve had a D-strategy s_E. Then, we used the strategy s_E in the game G, by applying s_E to the longest suffix of s which was a valid play in G.

For basic determinacy types, the result was a D-strategy. However, it is not so for persistent strategies: a previous visit in G could have fixed Eve's moves in some positions. To repair this problem, we use Theorem 8.22. Instead of using the strategy s_E for the whole game G, we use the respective strategy for the game where some moves (a finite number of them) are already fixed — Theorem 8.22 guarantees that such strategy exists. ∎

Chapter 9

Conclusion

As a conclusion, we recollect open problems which are related to the results presented in this thesis.

9.1 Closure under Union

We would like to know more closure properties of the class of half-positionally determined winning conditions. Specifically, our Conjecture 7.1 asks whether an union (finite or countable) of (finitely) half-positional conditions remains half-positional. It is known that an uncountable union does not need to be half-positional (Theorem 7.2). In many special cases it is known that unions of specific half-positional conditions are half-positional (Theorem 3.7, Chapter 7).

Specifically, we know that all XPS conditions have this property (Theorem 7.12), and that this class is closed under finite union. All half-positional winning conditions constructed in this thesis (except concave conditions – this is a property rather than a construction) fall in this class; it is possible that XPS captures all the reasons for a winning condition to be half-positional, and therefore all half-positional conditions are there.

9.2 ω-regular Conditions

In Theorem 6.10 we have shown that *finite* half-positional determinacy of winning conditions is decidable. We used the fact that if a winning condition is not half-positional, then there is a very simple arena witnessing it (Theorem 6.9); this fact was obtained via induction over the number of Eve's positions where she has a choice. However, what about *infinite* half-positional determinacy? In this case we can no longer use our inductive argument. One can easily create an infinite arena where applying the method used in proof of Theorem 6.9 leads to an arena where Eve no longer can win (after an infinite number of steps).

Also, the algorithm given in Theorem 6.10 is exponential, which is not satisfactory. It is possible that there is a simpler property which also answers whether a given ω-regular winning condition is (finitely) half-positional.

9.3 Types of Arenas

In section 2.5 we have introduced three types of arenas. We have shown examples of winning conditions which are half-positional when restricted to position-colored arenas, but not on all edge-colored arenas. We gave some arguments why we regard the broader classes of arenas as more natural when discussing positional determinacy.

The problem remains whether a winning condition which is half-positional with respect to edge-colored arenas has to be half-positional for all ϵ-arenas.

9.4 Chromatic Memory

Section 8.2 raises a problem about strategies which are allowed to use memory, but want to use as few memory states as possible. Is it always possible to create a memory of the smallest possible size which also has a nice property of being independent from the arena, i.e. a chromatic memory of size $\mathrm{mm}_X(W)$? We already have an algorithm for calculating $\mathrm{mm}^{\chi}_X(W)$ (Theorem 8.14), but not for $\mathrm{mm}_X(W)$ — a positive answer would mean that we don't need another one. This result could potentially simplify proofs of further results about strategies with memory.

9.5 Geometrical Conditions

The results in Chapter 5 do not cover all possible cases. We still do not know whether $WF(A)$ is finitely half-positionally determined for all co-convex sets A, and whether it is half-positionally determined for all co-convex open sets A. For A's which are unions of a finite number of half-spaces, e.g. $A = A_1 \cup A_2$, we cannot obtain half-positional determinacy via Theorem 5.9 and Theorem 7.10 (union of positional/ suspendable conditions), because this does not lead to $WF(A)$, but in general to a different set: $WF(A_1 \cup A_2)$ says that each cluster point is element of $A_1 \cup A_2$, and $WF(A_1) \cup WF(A_2)$ says that either each cluster point is an element of A_1 or each cluster point is an element of A_2.

9.6 Extensions

Another area of research is to extend our results to more general settings. There are several possible extensions.

One of them is examining *payoff mappings* or *preference relations* instead of winning conditions, which allow a game to have a wide spectrum of results instead of win or lose. See Section 2.6, and also [GZ04], [GZ05], [EM79].

We can also try to relax our requirement of prefix independence. If we hope for positional strategies, then the most important thing about prefix independence is that the past should not alter what is good for us in the future: if v_1w_1 is better than v_1w_2 for a player, then v_2w_2 cannot be better than v_2w_1 for $v_1,\ v_2,\ w_1,\ w_2 \in C^*$. We could also use a stronger version: w_1 is better than w_2 iff ww_1 is better than ww_2. *Monotone preference relations* are defined in [GZ05] in a similar way.

So far, we have either considered all possible arenas or restricted to finite arenas only. However, there are more examples of interesting classes of arenas. One example is push-down graphs, which are infinite, but have a finite representation. Another one is infinite arenas with finite branching. And of course, more research of position-colored arenas would be useful.

Another generalization is *stochastic games*. In addition to Eve's positions (where a "good" player decides) and Adam's positions (where a "bad" player decides), these games also allow random positions, where a move is decided randomly. In this setting we are interested in *optimal* strategies, which lead to the greatest possible probability of winning, or the greatest expected value of payoff in case of payoff mappings. Several new papers by Gimbert and Zielonka, e.g. [GZ07], explore this setting.

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Notation index

These tables present the commonly used notation. In each table a relevant page number is given for each notation.

The first table lists the usual names for arbitrary objects of given class. More objects of the same class are named by adding indexes or primes (e.g. strategies are named s, s', s_1 etc.).

k, l, m, n	integer	
α, β, γ, λ	ordinal	
c	color	13
W	winning condition	13
G	arena	13
X	player (Adam or Eve)	13
L	language of finite or infinite words	13
π	play	14
s	strategy	15
D	determinacy type	16
p	move	13
M	subset of Pos	21
u, v, w	positions or words (finite or infinite)	13
$\mathcal{M}$	memory	67

Names of specific winning conditions:

WB_S	Büchi condition, $WB_S = C^*(SC^*)^\omega$	24
WB'_S	co-Büchi condition, $WB'_S = C^*(C - S)^\omega$	24
$WF(A)$	universal geometrical condition	35
$WF'(A)$	existential geometrical condition	35
WM_A	monotonic condition	46
WP_n	parity condition over $\{0, \ldots, n\}$	25
WQ	example from Page 34	

Formal language and automata notation:

C	set of colors	13
a, **b**, **c**	default names of colors in examples	
L_1L_2	concatenation	13
L^*	Kleene iteration of L	13
L^ω	infinite iteration of L	13
$\|w\|$	length of the word w	13
$w_{\|n}$	first n letters of w	13
ϵ	an empty word	13
$v^{[\phi]}$	v if ϕ is true, ϵ otherwise	32
Q	automaton's set of states	44
q_I	automaton's initial state	44
δ	automaton's transition function	44
rank	parity automaton's rank function	44
L_A	language recognized by automaton A	44

Notation specific for games:

A, E	Adam and Eve	13
Pos	set of positions ($\mathrm{Pos} = \mathrm{Pos}_E \cup \mathrm{Pos}_A$)	13
Mov	set of moves	13
$v \xrightarrow{c} w$	a move from v to w of color c	13
source(p)	source (v) of a move p	14
target(p)	target (w) of a move p	14
rank(p)	color (c) of a move p	14
Play	set of all plays (infinite, ending on X's position)	14
Play_∞	set of infinite plays	14
Play_X	set of plays ending in X's position	14
source(π)	source (first position) of π	14
target(π)	target (last position) of π	14
rank(π)	sequence of colors in π	14
Win_X	X's winning set	15
$(\alpha_E, \alpha_A, \gamma)$	a determinacy type	16
$\mathrm{Next}_X(M)$	next move operator	21
$\mathrm{Attr}_X(M)$	attractor	21
$M[s]$	forward closure	21
$P(w)$	average color in w	35
$P_n(w)$	average color in $w_{\|n}$	35
Σ	suspension set	58
$\mathrm{mm}_X(W)$	minimum size of memory	68

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